

# EQUIVARIANT COHOMOLOGY OF WEIGHTED GRASSMANNIANS AND WEIGHTED SCHUBERT CLASSES

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**ABSTRACT.** In this paper, we study the  $wR$ -equivariant cohomology of the weighted Grassmannians  $wGr(d, n)$  introduced by Corti-Reid [4] where  $wR$  is the  $(n - 1)$ -dimensional torus that naturally acts on  $wGr(d, n)$ . We introduce the equivariant *weighted Schubert classes* and, after we show that they form a basis of the cohomology, we give an explicit formula for the structure constants with respect to this Schubert basis. We also find a particular rational basis  $\{wu_1, \dots, wu_n\}$  of  $\text{Lie}(wR)^*$  such that those structure constants are polynomials in  $wu_i$ 's with non-negative coefficients (up to a permutation on the weights). Furthermore, we find the relation between the factorial Schur functions and our equivariant weighted Schubert classes.

## 1. Introduction

The *weighted Grassmannian*  $wGr(d, n)$  introduced and studied by Corti-Reid [4], following the work of Grojnowski, is a projective variety with at worst orbifold singularity with a torus action. It is a generalization of the ordinary Grassmannian and is defined in a weighted projective space by the well-known *Plücker relations* as weighed homogeneous polynomials with appropriate weights. In this paper, we define the *weighted Schubert classes* and we show that they will form a basis of equivariant/non-equivariant cohomologies of  $wGr(d, n)$  over  $\mathbb{Q}$ -coefficients. Our main goal is to study the structure constants of the cohomology rings in terms of these weighted Schubert classes. The explicit formula of these structure constants for the weighted Grassmannian is cleverly derived from the Knutson-Tao's puzzle formula [16] for the ordinary Grassmannian, by detouring to the equivariant cohomology of the quasi-projective variety  $aPl(d, n)^\times$  defined in the affine space by the Plücker relations. We have found appropriate equivariant parameters in which our equivariant structure constants are polynomials with non-negative rational coefficients when the weights are non-decreasing (hence this implies that the structure constants of the ordinary cohomology are also non-negative). This is an analogue of the equivariant positivity proved by Graham [10], although we do not have the geometric or representation-theoretic interpretation of those parameters, while as, in [10], they are the simple roots in the character group of the maximal torus when we regard the flag varieties as homogeneous varieties.

Below we summarize our results in greater detail. Recall that the ordinary Grassmannian  $Gr(d, n)$  is the space of  $d$ -dimensional subspaces in the  $n$ -dimensional complex plane  $\mathbb{C}^n$ . It can be described as a non-singular projective variety of dimension  $d(n - d)$  defined by the well-known homogeneous polynomials (2.4), called the *Plücker relations*. It is embedded in the projective space  $\mathbb{P}(\mathbb{C}^{\binom{n}{d}})$  where  $\{n_d\} := \{\{\lambda_1, \dots, \lambda_d\} \mid 1 \leq \lambda_1 < \dots < \lambda_d \leq n\}$  and  $\mathbb{C}^{\binom{n}{d}}$  is the affine space of the plücker coordinates. Let  $aPl(d, n)$  be the affine variety in  $\mathbb{C}^{\binom{n}{d}}$  defined by the plücker

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relation and let  $\mathrm{aPl}(d, n)^\times := \mathrm{aPl}(d, n) - \{0\}$ . The  $n$ -dimensional complex torus  $T_{\mathbb{C}} := (\mathbb{C}^\times)^n$  naturally acts on  $\mathrm{aPl}(d, n)$  and  $\mathrm{aPl}(d, n)^\times$ , through the homomorphism

$$\rho : T_{\mathbb{C}} \rightarrow (\mathbb{C}^\times)^{\binom{n}{d}}, \quad (t_1, \dots, t_n) \mapsto (t_\lambda := t_{\lambda_1} \cdots t_{\lambda_d})_{\lambda \in \binom{n}{d}}.$$

The Grassmannian  $\mathrm{Gr}(d, n)$  is the quotient of  $\mathrm{aPl}(d, n)^\times$  by the diagonal torus  $D_{\mathbb{C}} \subset T_{\mathbb{C}}$  and hence  $\mathrm{Gr}(d, n)$  carries the residual action of the torus  $R_{\mathbb{C}} := T_{\mathbb{C}}/D_{\mathbb{C}}$ .

Following [4], we define the *weighted Grassmannian*  $\mathrm{wGr}(d, n)$  as the quotient of  $\mathrm{aPl}(d, n)^\times$  by the *locally free* action of a “twisted diagonal”  $\mathrm{w}D_{\mathbb{C}}$  in  $T_{\mathbb{C}}$ : for  $w := (w_1, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$  and  $a \in \mathbb{Z}_{\geq 1}$ , let

$$\mathrm{w}D_{\mathbb{C}} := \{(t^{dw_1+a}, \dots, t^{dw_n+a}) \in T_{\mathbb{C}} \mid t \in \mathbb{C}^\times\}.$$

The weighted Grassmannian is defined as

$$\mathrm{wGr}(d, n) := \mathrm{aPl}(d, n)^\times / \mathrm{w}D_{\mathbb{C}},$$

together with the residual action of the quotient torus  $\mathrm{w}R_{\mathbb{C}} := T_{\mathbb{C}}/\mathrm{w}D_{\mathbb{C}}$ . It is a projective variety with at worst orbifold singularities naturally embedded in the weighted projective space  $\mathbb{P}_w(\mathbb{C}^{\binom{n}{d}}) := (\mathbb{C}^{\binom{n}{d}} - \{0\})/\rho(\mathrm{w}D_{\mathbb{C}})$  with the weights

$$(w_\lambda := w_{\lambda_1} + \cdots + w_{\lambda_d} + a)_{\lambda \in \binom{n}{d}}.$$

In Section 2, we study the analogue of the usual *Schubert cell (Bruhat) decomposition* for  $\mathrm{aPl}(d, n)^\times$  and  $\mathrm{wGr}(d, n)$ . Indeed, there is a decomposition of  $\mathrm{aPl}(d, n)^\times$

$$\mathrm{aPl}(d, n)^\times = \coprod_{\lambda} \mathrm{a}\Omega_{\lambda}^{\circ}$$

lifted from the usual Bruhat decomposition  $\mathrm{Gr}(d, n) = \coprod_{\lambda} \Omega_{\lambda}^{\circ}$  by the quotient map  $\mathrm{aPl}(d, n)^\times \rightarrow \mathrm{Gr}(d, n)$ . It descends to a decomposition of  $\mathrm{wGr}(d, n)$  through the quotient map  $\mathrm{aPl}(d, n)^\times \rightarrow \mathrm{wGr}(d, n)$ :

$$\mathrm{wGr}(d, n) = \coprod_{\lambda} \mathrm{w}\Omega_{\lambda}^{\circ}.$$

We observe that the latter is a *quasi-cell decomposition*, i.e. each  $\mathrm{w}\Omega_{\lambda}^{\circ}$  is homeomorphic to a quotient of a complex plane by a finite group. By the standard arguments in algebraic topology, this allows us to show our first result (more technical versions of the claims will be in the main body of the paper).

**Proposition A** (Proposition 2.12, 2.13 below). *The rational cohomology  $H^*(\mathrm{wGr}(d, n))$  is concentrated in even degree. As a consequence, the equivariant cohomology  $H_{\mathrm{w}R}^*(\mathrm{wGr}(d, n))$  is a free module over  $H^*(B\mathrm{w}R)$  where  $\mathrm{w}R$  is the real torus in  $\mathrm{w}R_{\mathbb{C}}$ .*

*In this paper, all cohomologies are assumed to be over  $\mathbb{Q}$ -coefficients unless otherwise specified.*

We also use the decomposition to introduce the corresponding Schubert varieties in  $\mathrm{aPl}(d, n)^\times$  and  $\mathrm{wGr}(d, n)$  as the closures of the “cells”:

$$\mathrm{a}\Omega_{\lambda} := \overline{\mathrm{a}\Omega_{\lambda}^{\circ}} = \coprod_{\mu \geq \lambda} \mathrm{a}\Omega_{\mu}^{\circ}, \quad \mathrm{w}\Omega_{\lambda} := \overline{\mathrm{w}\Omega_{\lambda}^{\circ}} = \coprod_{\mu \geq \lambda} \mathrm{w}\Omega_{\mu}^{\circ}$$

where  $\geq$  is the usual Bruhat order defined in (2.1) and the second equality in each equation is shown in Proposition 2.7 and 2.8.

In Section 3, we explain the key isomorphisms among the equivariant cohomology rings of  $\mathrm{Gr}(d, n)$ ,  $\mathrm{aPl}(d, n)^\times$  and  $\mathrm{wGr}(d, n)$ . The claim seems fairly well-known and essentially follows

from the *Victoris-Begle mapping theorem*, however we give a careful proof at Section 3 since we haven't found one in literatures. More precisely, consider the natural maps of the Borel constructions of those spaces:

$$ER \times_R \mathrm{Gr}(d, n) \xleftarrow{h} ET \times_T \mathrm{aPl}(d, n)^\times \xrightarrow{wh} EwR \times_{wR} w\mathrm{Gr}(d, n),$$

where  $T, R$  and  $wR$  are the real tori in  $T_{\mathbb{C}}, R_{\mathbb{C}}$  and  $wR_{\mathbb{C}}$  respectively. Since the fibers of  $h$  and  $wh$  are the quotients of contractible spaces by finite groups, they induce isomorphisms on the rational equivariant cohomologies:

**Proposition B** (Proposition 3.1 below). *The pullback maps on the rational equivariant cohomologies*

$$H_R^*(\mathrm{Gr}(d, n)) \xrightarrow{h^*} H_T^*(\mathrm{aPl}(d, n)^\times) \xleftarrow{wh^*} H_{wR}^*(w\mathrm{Gr}(d, n))$$

are isomorphisms of rings over  $H^*(BR)$  and  $H^*(BwR)$  respectively.

Having these isomorphisms, we introduce the Schubert classes  $\tilde{a}\tilde{S}_\lambda$  in  $H_T^*(\mathrm{aPl}(d, n)^\times)$  to be the  $T$ -equivariant classes  $[\mathrm{a}\Omega_\lambda]_T$  associated the closed irreducible  $T$ -subvarieties  $\mathrm{a}\Omega_\lambda$  and define the *weighted Schubert classes*  $w\tilde{S}_\lambda$  in  $H_{wR}^*(w\mathrm{Gr}(d, n))$  by

$$w\tilde{S}_\lambda := (wh^*)^{-1}(\tilde{a}\tilde{S}_\lambda).$$

Note that the usual equivariant Schubert class  $\tilde{S}_\lambda$  in  $H_R^*(\mathrm{Gr}(d, n))$  considered in [16] coincides with  $(h^*)^{-1}(\tilde{a}\tilde{S}_\lambda)$  (see Corollary 4.6).

*Remark.* Following [7] and [18],  $H_T^*(\mathrm{aPl}(d, n)^\times)$  can be identified with the  $wR$ -equivariant cohomology of the quotient stack  $[\mathrm{aPl}(d, n)^\times / wD_{\mathbb{C}}]$  and the isomorphism  $wh^*$  is nothing but the identification of the (equivariant) rational cohomology rings of the weighted Grassmannian orbifold stack  $[\mathrm{aPl}(d, n)^\times / wD_{\mathbb{C}}]$  and its coarse moduli space  $w\mathrm{Gr}(d, n)$ . Under these identifications,  $\tilde{a}\tilde{S}_\lambda$  and  $w\tilde{S}_\lambda$  should be regarded as the class associated to the  $wR$ -invariant substack  $[\mathrm{a}\Omega_\lambda / wD_{\mathbb{C}}]$ . It should then coincide with the Poincaré dual of the cycle  $[w\Omega_\lambda]$  up to the multiplicity of the substack  $[\mathrm{a}\Omega_\lambda / wD_{\mathbb{C}}]$  in  $[\mathrm{aPl}(d, n)^\times / wD_{\mathbb{C}}]$ . We leave this aspect of the theory to elsewhere.

In Section 4, we work out explicitly the *GKM (Goresky-Kottwitz-Macpherson)* descriptions of  $H_T^*(\mathrm{aPl}(d, n)^\times)$  and  $H_{wR}^*(w\mathrm{Gr}(d, n))$ , following [9] and [20]. Namely we observe that there is the following commutative diagram of injective localization maps combined with the isomorphisms  $h^*$  and  $wh^*$ :

$$\begin{array}{ccc} H_R^*(\mathrm{Gr}(d, n)) & \longrightarrow & \bigoplus_\lambda H^*(BR) \\ h^* \downarrow \cong & & \cong \downarrow \\ H_T^*(\mathrm{aPl}(d, n)^\times) & \longrightarrow & \bigoplus_\lambda H^*(BT_\lambda) \\ wh^* \uparrow \cong & & \cong \uparrow \\ H_{wR}^*(w\mathrm{Gr}(d, n)) & \longrightarrow & \bigoplus_\lambda H^*(BwR) \end{array}$$

where  $T_\lambda$  is the kernel of  $T \rightarrow S^1, t \mapsto t_\lambda$ . The GKM descriptions of  $H_T^*(\mathrm{aPl}(d, n)^\times)$  and  $H_{wR}^*(w\mathrm{Gr}(d, n))$  are obtained in Proposition 4.2 and 4.3, from the well-known one for  $H_R^*(\mathrm{Gr}(d, n))$  by using the vertical isomorphisms. Furthermore, the upper triangularity of the image of  $\tilde{a}\tilde{S}_\lambda$  and  $w\tilde{S}_\lambda$  is given in Proposition 4.5 and 4.7, and as a consequence, we have

**Proposition C** (Proposition 4.8 below).  $\{\tilde{w}\tilde{S}_\lambda\}_\lambda$  is a basis of  $H_{\text{wR}}^*(\text{wGr}(d, n))$  as a module over  $H^*(B\text{wR})$ .

This allows us to define the structure constants  $\text{w}\tilde{c}_{\lambda\mu}^\nu$  of  $H_{\text{wR}}^*(\text{wGr}(d, n))$  by

$$\text{w}\tilde{S}_\lambda \cdot \text{w}\tilde{S}_\mu = \sum_\nu \text{w}\tilde{c}_{\lambda\mu}^\nu \text{w}\tilde{S}_\nu \quad \text{where } \text{w}\tilde{c}_{\lambda\mu}^\nu \in H^*(B\text{wR}).$$

Since  $T = (S^1)^n$ , we let  $H^*(BT) = \mathbb{Q}[y_1, \dots, y_n]$  where  $\{y_i\}$  is the standard basis of  $\text{Lie}(T)_{\mathbb{Z}}^*$ , the space of  $\mathbb{Z}$ -linear functions on the integral lattice of  $T$ . Consider the subspace of  $\text{Lie}(T)_{\mathbb{Z}}^* \otimes \mathbb{Q}$  generated by

$$\text{w}u_i := (y_{i+1} - y_i) - \frac{w_{i+1} - w_i}{w_{\text{id}}} y_{\text{id}} \quad \text{for } i = 1, \dots, n-1,$$

where  $\text{id} \in \{d\}$  is the unique minimum element in the Bruhat order and  $y_\lambda := y_{\lambda_1} + \dots + y_{\lambda_d}$ . By the surjection  $T \rightarrow \text{wR}$ , we can regard  $H^*(B\text{wR})$  as a subring of  $H^*(BT)$  and identify

$$H^*(B\text{wR}) = \mathbb{Q}[\text{w}u_1, \dots, \text{w}u_{n-1}] \subset H^*(BT).$$

Under this identification, we prove, in Section 5, the following equivariant positivity:

**Theorem D** (Theorem 5.4 below). *If  $w_1 \leq w_2 \leq \dots \leq w_n$ , then  $\text{w}\tilde{c}_{\lambda\mu}^\nu$  is a polynomial in  $\text{w}u_1, \dots, \text{w}u_{n-1}$  with non-negative rational coefficients.*

This positivity is actually based on the explicit formula of  $\text{w}\tilde{c}_{\lambda\mu}^\nu$  in terms of the equivariant puzzles introduced by Knutson-Tao [16]. For every puzzle, we choose a total order on the set of equivariant pieces once and for all. Let  $P$  be a puzzle satisfying  $\partial P = \Delta_{\lambda\mu}^\nu$ . Let  $p$  be an equivariant piece in  $P$ , whose weight is  $\text{wt}(p) := y_j - y_i, j > i$ , i.e.  $p$  pokes out the  $i$ -th and  $j$ -th place on the south side of  $P$ . For each  $\xi \in \{d\}$ , let

$$\text{wt}^\xi(p) := (y_j - y_i) - \frac{w(p)}{w_\xi} y_\xi \in \mathbb{Q}[\text{wR}^*], \quad \text{where } w(p) := w_j - w_i.$$

Let  $(p_1, \dots, p_r)$  be the set of equivariant pieces in  $P$  and  $\xi^k \rightarrow \dots \rightarrow \xi^0$  a covering sequence in  $\{d\}$  with  $k \leq r$ , i.e.  $\xi^i \geq \xi^{i-1}$  and  $l(\xi^i) = l(\xi^{i-1}) + 1$ . We then introduce a notation

$$\begin{aligned} \text{wt}^{\xi^0, \dots, \xi^k}(P) &:= \sum_{1 \leq i_1 < \dots < i_k \leq r} \\ &\text{wt}^{\xi^0}(p_1) \cdots \text{wt}^{\xi^0}(p_{i_1-1}) \cdot \frac{w(p_{i_1})}{w_{\xi^0}} \cdot \text{wt}^{\xi^1}(p_{i_1+1}) \cdots \text{wt}^{\xi^1}(p_{i_2-1}) \cdot \frac{w(p_{i_2})}{w_{\eta^1}} \\ &\quad \cdot \text{wt}^{\xi^2}(p_{i_2+1}) \cdots \text{wt}^{\eta^{k-1}}(p_{i_k-1}) \cdot \frac{w(p_{i_k})}{w_{\xi^{k-1}}} \cdot \text{wt}^{\xi^k}(p_{i_k+1}) \cdots \text{wt}^{\xi^k}(p_r). \end{aligned}$$

Now we arrive at the manifestly positive formula for  $\text{w}\tilde{c}_{\lambda\mu}^\nu$ :

**Theorem E** (Theorem 5.3 below).

$$\text{w}\tilde{c}_{\lambda\mu}^\nu = \left( \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\nu}} \text{wt}^\nu(P) \right) + \sum_{\substack{\nu \rightarrow \nu^1 \rightarrow \dots \rightarrow \nu^k \geq \lambda, \mu}} \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{\lambda\mu}^{\nu^k}}} \text{wt}^{\nu^k, \dots, \nu^1, \nu}(Q).$$

Remark that the  $w$ -weights in this expression depend on the orders of the equivariant pieces in  $P$  and  $Q$ , but  $\text{w}\tilde{c}_{\lambda\mu}^\nu$  doesn't depend on the order. The positivity follows from Proposition

5.5 which states that  $\text{wt}^\nu(P)$  and  $\text{wt}^{\nu^k, \dots, \nu^1, \nu}(Q)$  are polynomials in  $wu_i$ 's with non-negative coefficients when  $w_1 \leq \dots \leq w_n$ .

We derive this formula from the Knutson-Tao's formula for the structure constants  $\tilde{c}_{\lambda\mu}^\nu$  of  $H_R^*(\text{Gr}(d, n))$  through the isomorphisms  $h^*$  and  $wh^*$  in Proposition B. Namely, by the isomorphism  $h^*$ , the Knutson-Tao's formula translates to

$$a\tilde{S}_\lambda a\tilde{S}_\mu = \sum_{\eta \geq \lambda, \mu} \overbrace{\sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\eta}}^{\tilde{c}_{\lambda\mu}^\eta} \text{wt}(p_r) \cdots \text{wt}(p_1) a\tilde{S}_\eta \quad \text{in } H_T^*(a\text{Pl}(d, n)^\times),$$

Theorem E is obtained from this, by inductively applying the following formula in  $H_T^*(a\text{Pl}(d, n)^\times)$ , which is essentially equivalent to the *equivariant Pieri-rule* in  $H_R^*(\text{Gr}(d, n))$ :

**Proposition F** (Proposition 5.2 below). *Let  $p$  be an equivariant piece of a puzzle. Then*

$$\text{wt}(p)a\tilde{S}_\nu = \text{wt}^\nu(p)a\tilde{S}_\nu + \sum_{\nu' \rightarrow \nu} \frac{w(p)}{w_\nu} a\tilde{S}_{\nu'}.$$

We define the *non-equivariant weighted Schubert classes*  $wS_\lambda$  in the ordinary cohomology  $H^*(w\text{Gr}(d, n))$  as the image of  $w\tilde{S}_\lambda$  under the projection  $H_{wR}^*(w\text{Gr}(d, n)) \rightarrow H^*(w\text{Gr}(d, n))$ . Under the natural isomorphism  $H^*(w\text{Gr}(d, n)) \cong H_{wD}^*(a\text{Pl}(d, n)^\times)$ ,  $wS_\lambda$  corresponds to the  $wD$ -equivariant cohomology class  $[a\Omega_\lambda]_{wD}$  associated to the invariant subvariety  $a\Omega_\lambda$ . Then the ordinary structure constants  $wc_{\lambda\mu}^\nu$  are given by the non-equivariant limit  $wu_1 = \dots = wu_{n-1} = 0$ . Thus we obtain the formula for  $wc_{\lambda\mu}^\nu$  in terms of the structure constants  $c_{\lambda\mu}^\nu$  of  $H^*(\text{Gr}(d, n))$  computed in [17] and  $\tilde{c}_{\lambda\mu}^\nu$  computed in [16] as a polynomial in  $u_i := y_{i+1} - y_i$ :

**Corollary G** (Corollary 5.6 below)

$$wc_{\lambda\mu}^\nu = c_{\lambda\mu}^\nu + \sum_{\substack{\nu \rightarrow \nu^1 \rightarrow \dots \rightarrow \nu^k \geq \lambda, \mu}} \frac{\tilde{c}_{\lambda\mu}^{\nu^k}(u_i = w_{i+1} - w_i, i = 1, \dots, n-1)}{w_{\nu^1} \cdots w_{\nu^k}},$$

if  $l(\lambda) + l(\mu) = l(\nu)$  and is 0 otherwise. If  $w_1 \leq w_2 \leq \dots \leq w_n$ ,  $wc_{\lambda\mu}^\nu$  is non-negative for all  $\lambda, \mu, \nu \in \left\{\frac{n}{d}\right\}$ .

*Remark.* We can apply a permutation to the index  $(1, \dots, n)$  so that the weights  $\{w_1, \dots, w_n\}$  are non-decreasing without changing the space  $w\text{Gr}(d, n)$ . We can always find a Schubert basis that satisfies the positivity.

We conclude Section 5 by including the *equivariant weighted Pieri rule* (Lemma 5.8) and also the examples of  $w\text{Gr}(1, n)$  and  $w\text{Gr}(2, 4)$ . The space  $w\text{Gr}(1, n)$  is the well-known *weighted Projective space* and its integral cohomology are first studied by Kawasaki [13] and its equivariant cohomology by Bahri-Franz-Ray[2] and Tymoczko [24]. We discuss the relation of our Schubert basis to Kawasaki's basis over  $\mathbb{Z}$ -coefficients at Example 5.11. For  $w\text{Gr}(2, 4)$ , here are a few examples of the structure constants:

$$w\tilde{S}_{23}w\tilde{S}_{14} = \left((y_4 - y_1) - (w_4 - w_1)\frac{y_{13}}{w_{13}}\right)w\tilde{S}_{13} + \frac{w_4 - w_1}{w_{13}}w\tilde{S}_{12},$$

$$\begin{aligned}
w\tilde{S}_{23}w\tilde{S}_{23} &= \left( (y_4 - y_2) - (w_4 - w_2)\frac{y_{23}}{w_{23}} \right) \left( (y_4 - y_3) - (w_4 - w_3)\frac{y_{23}}{w_{23}} \right) w\tilde{S}_{23} \\
&\quad + \left( \frac{w_4 - w_3}{w_{23}} \left( (y_4 - y_2) - (w_4 - w_2)\frac{y_{23}}{w_{23}} \right) \right. \\
&\quad \quad \left. + \frac{w_{34}}{w_{23}} \left( (y_4 - y_3) - (w_4 - w_3)\frac{y_{13}}{w_{13}} \right) \right) w\tilde{S}_{13} \\
&\quad + \left( 1 + \frac{w_4 - w_3}{w_{13}} \frac{w_{34}}{w_{23}} \right) w\tilde{S}_{12}.
\end{aligned}$$

As you can see, with the help of Proposition 5.5, the assumption  $w_1 \leq \dots \leq w_4$  makes sure that the coefficients are positive.

Finally, in Section 6, we study the relation between the *factorial Schur functions*  $s_\lambda(x|a)$  and our equivariant weighted Schubert classes  $w\tilde{S}_\lambda$ . Namely we generalize [16, Lemma, Section 6] (see also [22], [21, Theorem 2.1]) that describes the restriction of  $\tilde{S}_\lambda$  to a fixed point by a specialization of the corresponding  $s_\lambda(x|a)$ .

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## 2. Weighted Grassmannians and Weighted Schubert Varieties

In this section, we recall the definition of the *weighted Grassmannian*  $w\text{Gr}(d, n)$ , following [4]. We study the coordinate charts and obtain a *quasi-cell decomposition* which generalizes the usual Schubert cell decomposition of the ordinary Grassmannian  $\text{Gr}(d, n)$ . This allows us to define the *weighted Schubert varieties* by taking the closure of each cell and also as a consequence, we show that the odd degree classes of the rational cohomology of  $w\text{Gr}(d, n)$  vanish.

For positive integers  $d$  and  $n$  such that  $d < n$ , let  $[n] := \{1, \dots, n\}$ , and

$$\{d\}^n := \{\lambda \subset [n] \mid |\lambda| = d\}.$$

We denote the elements of  $\lambda$  by  $\lambda_1, \dots, \lambda_d$  where  $\lambda_1 < \dots < \lambda_d$ . For  $\lambda, \mu \in \{d\}^n$ , we define the *Bruhat order* by:  $\lambda \leq \mu$  if

$$(2.1) \quad \lambda_i \geq \mu_i \text{ for all } i = 1, \dots, d.$$

We define the *lexicographic order* by:  $\lambda <_{\text{lex}} \mu$  if there exists an integer  $1 \leq j \leq d$  such that

$$(2.2) \quad \lambda_i = \mu_i \text{ for all } i < j, \text{ and } \lambda_j < \mu_j.$$

An *inversion*  $(k, l)$  of  $\lambda$  is a pair of  $k \in \lambda$  and  $l \notin \lambda$  such that  $k < l$ . Let  $\text{inv}(\lambda)$  be the set of all inversions of  $\lambda$ . The *length*  $l(\lambda)$  of  $\lambda$  is defined to be the cardinality of  $\text{inv}(\lambda)$ . For each  $(k, l) \in \text{inv}(\lambda)$ , let  $(k, l)\lambda$  be the element of  $\{d\}^n$  obtained by replacing  $k$  in  $\lambda$  by  $l$ . Let

$$[\lambda] := \{\mu \in \{d\}^n \mid |\lambda \cap \mu| = d - 1\}.$$

Then  $[\lambda] = [\lambda]_+ \amalg [\lambda]_-$  where  $[\lambda]_- := \{\mu \in [\lambda] \mid \mu \leq \lambda\}$  and  $[\lambda]_+ := \{\mu \in [\lambda] \mid \mu \geq \lambda\}$ . Note that there is a bijection  $\text{inv}(\lambda) \cong [\lambda]_-$ , sending  $(k, l)$  to  $(k, l)\lambda$ . We say that  $\lambda$  *covers*  $\mu$  if  $\mu \in [\lambda]_-$  and  $l(\lambda) = l(\mu) + 1$ , and denote  $\lambda \rightarrow \mu$ .

**2.1. The weighted Grassmannian.** Let  $\mathbb{C}^n$  be the complex  $n$ -plane with the standard basis  $\{e_i, i \in [n]\}$  and  $\bigwedge^d \mathbb{C}^n$  its  $d$ -th exterior product with the induced basis

$$\{e_\lambda := e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d}, \lambda \in \{n\}_d\}.$$

We identify  $\bigwedge^d \mathbb{C}^n$  with the coordinate space  $\mathbb{C}^{\{n\}_d}$  where each  $x \in \bigwedge^d \mathbb{C}^n$  corresponds to the coordinate vector  $(x_\lambda)_{\lambda \in \{n\}_d}$  with respect to  $e_\lambda$ 's. We also denote each coordinate  $x_\lambda$  by  $x(\lambda_1 \cdots \lambda_n)$ .

Let  $T_{\mathbb{C}} := (\mathbb{C}^\times)^n$  and  $(\mathbb{C}^\times)^{\{n\}_d}$  be the complex tori acting canonically on  $\mathbb{C}^n$  and  $\mathbb{C}^{\{n\}_d}$  respectively. Consider the following  $T_{\mathbb{C}}$ -equivariant map

$$\bigwedge^d : \overbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}^{d \text{ times}} \rightarrow \bigwedge^d \mathbb{C}^n \quad ; \quad (z_1, \dots, z_d) \mapsto z_1 \wedge \cdots \wedge z_d$$

where  $T_{\mathbb{C}}$  acts on the domain diagonally and, to the target through the map

$$(2.3) \quad \rho : T_{\mathbb{C}} \rightarrow (\mathbb{C}^\times)^{\{n\}_d} \quad ; \quad t = (t_1, \dots, t_n) \mapsto (t_\lambda := \prod_{l \in \lambda} t_l)_{\lambda \in \{n\}_d}.$$

Let  $\text{aPl}(d, n)$  be the image of  $\bigwedge^d$  which is  $T_{\mathbb{C}}$ -invariant.

Let  $w := (w_1, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$  and  $a \in \mathbb{Z}_{\geq 1}$ . We introduce

$$\text{wD}_{\mathbb{C}} := \{(t^{dw_1+a}, \dots, t^{dw_n+a}) \in T_{\mathbb{C}} \mid t \in \mathbb{C}^\times\} \quad \text{and} \quad \text{wR}_{\mathbb{C}} := T_{\mathbb{C}}/\text{wD}_{\mathbb{C}}.$$

Note that

$$\rho(\text{wD}_{\mathbb{C}}) = \left\{ (t^{w_\lambda})_\lambda \in (\mathbb{C}^\times)^{\{n\}_d} \mid t \in \mathbb{C}^\times \right\}, \quad \text{where } w_\lambda := a + \sum_{l \in \lambda} w_l.$$

In the case when  $w = (0, \dots, 0)$  and  $a = 1$ , we write  $D_{\mathbb{C}}$  for the diagonal in  $\mathbb{C}^{\{n\}_d}$  and  $R_{\mathbb{C}} = T_{\mathbb{C}}/D_{\mathbb{C}}$ . We denote the corresponding compact real tori in the complex tori by  $T, \text{wR}, \text{wD}, R$  and  $D$  respectively.

**Definition 2.1** (Corti-Reid [4]). Let  $\text{aPl}(d, n)^\times := \text{aPl}(d, n) - \{0\}$ . The *weighted Grassmannian*  $\text{wGr}(d, n)$  is the projective variety with at worst orbifold singularities, given by

$$\text{wGr}(d, n) := \text{aPl}(d, n)^\times / \text{wD}_{\mathbb{C}}.$$

The quotient torus  $\text{wR}_{\mathbb{C}}$  acts on  $\text{wGr}(d, n)$ . The ordinary Grassmannian  $\text{Gr}(d, n)$  is the special case when  $w_1 = \cdots = w_n = 0$  and  $a = 1$ , i.e.  $\text{Gr}(d, n) = \text{aPl}(d, n)^\times / D_{\mathbb{C}}$ .

**Remark 2.2.** In [4], the  $\mathbb{C}^\times$ -action which defines  $\text{wGr}(d, n)$  as a quotient of  $\text{aPl}(d, n)^\times$  is actually given by the map  $\mathbb{C}^\times \rightarrow \rho(\text{wD}_{\mathbb{C}})$ ,  $t \mapsto (t^{w_\lambda})_\lambda$ , but obviously it defines the same algebraic variety.

**2.2. The Charts for  $\text{aPl}(d, n)^\times$  and  $\text{wGr}(d, n)$ .** Extend the notation  $x(l_1, \dots, l_d)$  to any (not necessarily increasing) sequence  $(l_1, \dots, l_d)$  of integers in  $[n]$  by the rule

$$x(l_1, \dots, l_p, l_{p+1}, \dots, l_d) = -x(l_1, \dots, l_{p+1}, l_p, \dots, l_d)$$

for any integer  $1 \leq p \leq d-1$ . It is known that  $\text{aPl}(d, n)^\times$  is a non-singular quasi-projective variety in  $\mathbb{C}^{\{n\}_d} - \{0\}$  defined by of the *Plücker relations* (c.f. [15]): for any sequence of integers  $1 \leq j_1, \dots, j_{d-1}, l_1, \dots, l_{d+1} \leq n$ ,

$$(2.4) \quad \sum_{i=1}^{d+1} (-1)^{i-1} x(j_1, \dots, j_{d-1}, l_i) x(l_1, \dots, \check{l}_i, \dots, l_{d+1}) = 0.$$

Consider the following  $T_{\mathbb{C}}$ -stable open neighborhood of  $e_{\lambda}$  in  $\mathrm{aPl}(d, n)^{\times}$ :

$$\mathrm{a}U^{\lambda} := \{x \in \mathrm{aPl}(d, n)^{\times} \mid x_{\lambda} \neq 0\}.$$

It is clear that  $\mathrm{aPl}(d, n)^{\times}$  is covered by  $\mathrm{a}U^{\lambda}$ 's, and moreover we have the natural  $T_{\mathbb{C}}$ -equivariant coordinates on each  $\mathrm{a}U^{\lambda}$ . Let  $\mathbb{C}^{[\lambda]}$  be the subspace of  $\mathbb{C}^{\{d\}^n}$  corresponding to the subspace generated by  $\{e_{\mu}, \mu \in [\lambda]\}$  and consider the natural projection

$$(2.5) \quad \psi_{\lambda} : \mathrm{a}U^{\lambda} \rightarrow \mathbb{C}^{\times} \times \mathbb{C}^{[\lambda]} \quad ; \quad x \mapsto (x_{\lambda}, (x_{\mu})_{\mu \in [\lambda]}).$$

This is a  $T_{\mathbb{C}}$ -equivariant homeomorphism where  $T_{\mathbb{C}}$  acts on the target through the map

$$\rho_{\lambda} : T_{\mathbb{C}} \rightarrow (\mathbb{C}^{\times})^{\{d\}^n} \rightarrow \mathbb{C}^{\times} \times (\mathbb{C}^{\times})^{[\lambda]} \quad ; \quad t \mapsto (t_{\lambda}, (t_{\mu})_{\mu \in [\lambda]}).$$

Indeed, the inverse of  $\psi_{\lambda}$  is constructed as follows (c.f. [15, p.1065]). For each  $i = 1, \dots, d$  and  $l \in [n]$ , let

$$p_i(l) = \frac{x(\lambda_1 \cdots \lambda_{i-1} l \lambda_{i+1} \cdots \lambda_d)}{x_{\lambda}}$$

and consider the vectors  $p_i := \sum_{l=1}^n p_i(l) e_l \in \mathbb{C}^n$ . The numerator of each coefficient is  $\pm x_{\mu}$  with  $\mu \in [\lambda]$  if  $l \in [\lambda] \setminus \{\lambda_i\}$ ,  $x_{\lambda}$  if  $l = \lambda_i$ , and zero otherwise. Thus we can define  $\psi_{\lambda}$  by assigning  $y := x_{\lambda} p_1 \wedge \cdots \wedge p_d$  to each  $\tilde{x} := (x_{\lambda}, (x_{\mu})_{\mu \in [\lambda]}) \in \mathbb{C}^{\times} \times \mathbb{C}^{[\lambda]}$ . It is straightforward to check that  $y_{\mu} = x_{\mu}$  for all  $\mu \in [\lambda]$  and  $y_{\lambda} = x_{\lambda}$ , i.e.  $\psi_{\lambda}(y) = \tilde{x}$ .

Passing to the quotient, we obtain the natural  $\mathrm{w}R_{\mathbb{C}}$ -equivariant affine charts of  $\mathrm{wGr}(d, n)$ . Let

$$\mathrm{w}U^{\lambda} := \mathrm{a}U^{\lambda} / \mathrm{w}D_{\mathbb{C}}.$$

Then  $\psi_{\lambda}$  induces a homeomorphism

$$\overline{\psi}_{\lambda} : \mathrm{w}U^{\lambda} \xrightarrow{\cong} (\mathbb{C}^{\times} \times \mathbb{C}^{[\lambda]}) / \rho_{\lambda}(\mathrm{w}D_{\mathbb{C}}) \cong \mathbb{C}^{[\lambda]} / G_{\lambda}$$

where  $G_{\lambda}$  is a finite cyclic subgroup of  $(\mathbb{C}^{\times})^{[\lambda]}$  given by

$$G_{\lambda} = \left\{ (t^{w_{\mu}})_{\mu \in [\lambda]} \in (\mathbb{C}^{\times})^{[\lambda]} \mid t \in \mathbb{C}^{\times} \text{ and } t^{w_{\lambda}} = 1 \right\}.$$

**2.3. The Schubert cell decompositions and Schubert varieties.** Consider the  $(\mathbb{C}^{\times})^{\{d\}^n}$ -invariant decomposition of  $\mathbb{C}^{\{d\}^n} - \{0\}$

$$\mathbb{C}^{\{d\}^n} - \{0\} := \coprod_{\lambda \in \{d\}^n} C_{\lambda}$$

where

$$C_{\lambda} := \left\{ x \in \mathbb{C}^{\{d\}^n} - \{0\} \mid x_{\lambda} \neq 0 \text{ and } x_{\mu} = 0 \text{ for all } \mu >_{lex} \lambda \right\}.$$

By restricting the above decomposition to  $\mathrm{aPl}(d, n)^{\times}$ , there is the  $T_{\mathbb{C}}$ -invariant decomposition

$$(2.6) \quad \mathrm{aPl}(d, n)^{\times} = \coprod_{\lambda \in \{d\}^n} \mathrm{a}\Omega_{\lambda}^{\circ} \quad \text{where} \quad \mathrm{a}\Omega_{\lambda}^{\circ} := \mathrm{aPl}(d, n)^{\times} \cap C_{\lambda}.$$

Since  $\mathrm{aPl}(d, n)^{\times} \cap C_{\lambda} \subset \mathrm{a}U^{\lambda}$ , we have  $\mathrm{a}\Omega_{\lambda}^{\circ} = \mathrm{a}U^{\lambda} \cap C_{\lambda}$ . The following lemma helps us to describe the image of  $\mathrm{a}\Omega_{\lambda}^{\circ}$  under the chart  $\psi_{\lambda}$  (Cor 2.4).

**Lemma 2.3.** *Let  $x \in \mathrm{a}U^{\lambda}$ . The following are equivalent:*

- (i)  $x_{\mu} = 0$  for all  $\mu \in [\lambda]_{-}$ .
- (ii)  $x_{\mu} = 0$  for all  $\mu >_{lex} \lambda$  (i.e.  $x \in \mathrm{a}\Omega_{\lambda}^{\circ}$ ).
- (iii)  $x_{\mu} = 0$  for all  $\mu \not\leq \lambda$ .



*Proof.* Since we have the implications  $\nu \in [\lambda]_- \Rightarrow \nu >_{lex} \lambda \Rightarrow \nu \not\geq \lambda$ , it is clear that (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i). We prove that (i) implies (iii). Assume (i) and let  $\nu \not\geq \lambda$ . We use induction on the number  $k := d - |\nu \cap \lambda|$ . If  $k = 1$ ,  $\nu \not\geq \lambda$  implies  $\nu \in [\lambda]_-$ . Thus  $x_\nu = 0$ . In general, choose an integer  $1 \leq s \leq d$  such that  $\nu_s \notin \lambda$ . For  $1 \leq i \leq d$  such that  $\lambda_i \notin \nu$ , let  $\nu^{(i)} := (\nu \setminus \{\nu_s\}) \cup \{\lambda_i\}$  and  $\lambda^{(i)} := (\lambda \setminus \{\lambda_i\}) \cup \{\nu_s\}$  in  $\{d\}$ . Then we claim that

$$(2.7) \quad \lambda \not\geq \lambda^{(i)} \text{ or } \lambda \not\geq \nu^{(i)}.$$

Indeed,  $\lambda \leq \lambda^{(i)}$  implies  $\nu_s < \lambda_i$ , i.e.  $\nu^{(i)} \leq \nu$ . Therefore, together with  $\lambda \leq \nu^{(i)}$ , it implies  $\nu \geq \lambda$ . Thus the negation of (2.7) leads to a contradiction. Now, consider the Plücker relation for the sequences  $\nu_1, \dots, \nu_s, \dots, \nu_d$  and  $\lambda_1, \dots, \lambda_d, \nu_s$

$$(2.8) \quad x_\nu x_\lambda = \sum_{\substack{1 \leq i \leq d \\ \lambda_i \notin \nu}} \pm x_{\nu^{(i)}} x_{\lambda^{(i)}}$$

where the signs are chosen appropriately according to (2.4). Since we have  $d - |\nu^{(i)} \cap \lambda| < d - |\nu \cap \lambda|$  and  $d - |\lambda^{(i)} \cap \lambda| = 1$ , the induction hypothesis implies that  $x_{\nu^{(i)}} = 0$  or  $x_{\lambda^{(i)}} = 0$  by (2.7). Therefore (2.8) becomes  $x_\nu x_\lambda = 0$ . Since  $x_\lambda \neq 0$  by  $x \in aU^\lambda$ , we have  $x_\nu = 0$ .  $\square$

**Corollary 2.4.** *Under the chart  $\psi_\lambda$ , we have  $a\Omega_\lambda^\circ \cong \mathbb{C}^\times \times \mathbb{C}^{[\lambda]_+} \times \{0\}^{[\lambda]_-}$ .*

Since the decomposition (2.6) is  $T_{\mathbb{C}}$ -invariants, it descends to the quotient  $wGr(d, n)$  and gives the  $wR_{\mathbb{C}}$ -invariant decomposition. We call this decomposition a *quasi-cell decomposition* because each “cell” is actually homeomorphic to a Euclidean space modulo a finite group.

**Proposition 2.5.**

$$wGr(d, n) = \coprod_{\lambda \in \{d\}^n} w\Omega_\lambda^\circ \quad \text{where} \quad w\Omega_\lambda^\circ := a\Omega_\lambda^\circ / wD_{\mathbb{C}}.$$

Under the chart  $\bar{\psi}_\lambda$ ,  $w\Omega_\lambda^\circ \cong \mathbb{C}^{[\lambda]_+} / G_\lambda$ .

**Definition 2.6.** For each  $\lambda \in \{d\}^n$ , we define the Schubert varieties in  $aPl(d, n)^\times$  and  $wGr(d, n)$  as Euclidean closures of  $a\Omega_\lambda^\circ$  and  $w\Omega_\lambda^\circ$  respectively, i.e.

$$a\Omega_\lambda := \overline{a\Omega_\lambda^\circ} \quad \text{and} \quad w\Omega_\lambda := \overline{w\Omega_\lambda^\circ}.$$

We will call  $w\Omega_\lambda$  the *weighted Schubert variety* corresponding to  $\lambda$ .

The next proposition seems well-known but for the sake of completeness we will give a proof in the appendix.

**Proposition 2.7.**

$$a\Omega_\lambda = \coprod_{\mu \geq \lambda} a\Omega_\mu^\circ.$$

The following proposition is a corollary.

**Proposition 2.8.**

$$w\Omega_\lambda = \coprod_{\mu \geq \lambda} w\Omega_\mu^\circ$$

*Proof.* It is clear from Proposition 2.7 that  $w\Omega_\lambda \supset \coprod_{\lambda \leq \mu} w\Omega_\mu^\circ$ . Let  $[x] \in w\Omega_\lambda$ . There exists a sequence  $\{[x^i]\}_{i=0}^\infty \subset w\Omega_\lambda^\circ$  such that  $[x^i] \rightarrow [x]$  as  $i \rightarrow \infty$ . Lemma 2.3 shows  $x_\eta^i = 0$  for all  $\eta \not\geq \lambda$ . This implies that  $x_\eta = 0$  for all  $\eta \not\geq \lambda$ . Thus  $x$  must lie in  $w\Omega_\mu^\circ$  for some  $\mu \geq \lambda$ .  $\square$

As corollaries, we can also write  $a\Omega_\lambda$  and  $w\Omega_\lambda$  explicitly as subvarieties of  $aPl(d, n)^\times$  and  $wGr(d, n)$  respectively.

**Corollary 2.9.**  $\mathfrak{a}\Omega_\lambda = \{x \in \mathfrak{a}\mathrm{Pl}(d, n)^\times \mid x_\nu = 0 \text{ for all } \nu \not\geq \lambda\}$ . In particular, the complex codimension of  $\mathfrak{a}\Omega_\lambda$  in  $\mathfrak{a}\mathrm{Pl}(d, n)^\times$  is the length  $l(\lambda)$  of  $\lambda$ .

*Proof.* Let  $x \in \mathfrak{a}\Omega_\lambda$ . By Proposition 2.7,  $x$  is contained in  $\mathfrak{a}\Omega_\mu^\circ$  for some  $\mu \geq \lambda$ . If  $\nu \not\geq \lambda$ , then  $\nu \not\geq \mu$  so that  $x_\nu = 0$  by Lemma 2.3. On the other hand, suppose that  $x$  is in the RHS. Then  $x \notin \mathfrak{a}\Omega_\nu$  for all  $\nu \not\geq \lambda$ . Therefore  $x \in \coprod_{\mu \geq \lambda} \mathfrak{a}\Omega_\mu^\circ = \mathfrak{a}\Omega_\lambda$ . Finally  $\dim \mathfrak{a}\Omega_\lambda = \dim \mathfrak{a}\Omega_\lambda^\circ = d(n-d) - l(\lambda)$  by Corollary 2.4.  $\square$

The previous corollary immediately implies the following.

**Corollary 2.10.**  $\mathfrak{w}\Omega_\lambda = \{[x] \in \mathfrak{w}\mathrm{Gr}(d, n) \mid x_\nu = 0 \text{ for all } \nu \not\geq \lambda\}$ . In particular, the codimension of  $\mathfrak{w}\Omega_\lambda^\circ$  in  $\mathfrak{w}\mathrm{Gr}(d, n)$  is the length  $l(\lambda)$  of  $\lambda$ .

**Remark 2.11.** The varieties  $\mathfrak{a}\Omega_\lambda$  is irreducible in  $\mathfrak{a}\mathrm{Pl}(d, n)^\times$  since it is a closure of  $\Omega_\lambda^\circ$ . From this, it also follows that  $\mathfrak{w}\Omega_\lambda$  is an irreducible variety.

**2.4. Vanishing of the odd degree.** The quasi-cell decomposition in Proposition 2.5 allows us to show that the odd degree of the rational singular cohomology  $H^*(\mathfrak{w}\mathrm{Gr}(d, n))$  vanishes. As a consequence, the Serre spectral sequence for the fibration  $E\mathfrak{w}R \times_{\mathfrak{w}R} \mathfrak{w}\mathrm{Gr}(d, n) \rightarrow B\mathfrak{w}R$  degenerates at  $E_2$ -stage and the  $\mathfrak{w}R$ -equivariant cohomology is free over  $H^*(B\mathfrak{w}R)$ . In this paper, all cohomologies are assumed to be over  $\mathbb{Q}$ -coefficients unless otherwise specified.

**Proposition 2.12.**

$$H^i(\mathfrak{w}\mathrm{Gr}(d, n)) \cong \begin{cases} \bigoplus_{2l(\lambda)=i} \mathbb{Q} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

*Proof.* The argument of Appendix B in [8] can be applied to the quasi-cell decomposition, and we obtain

$$(2.9) \quad \overline{H}_i(\mathfrak{w}\mathrm{Gr}(d, n)(d, n)) \cong \bigoplus_{\dim \mathfrak{w}\mathrm{Gr}(d, n) - 2l(\lambda) = i} \overline{H}_i(\mathfrak{w}\Omega_\lambda^\circ)$$

where  $\overline{H}_*$  is the rational Borel-Moore homology. The following is a standard fact (For a proof, see Section 7.2 in Appendix):

$$(2.10) \quad \overline{H}_i(\mathfrak{w}\Omega_\lambda^\circ) \cong \overline{H}_i(\mathbb{C}^{[\lambda]-}/G_\lambda) \cong \begin{cases} \mathbb{Q} & \text{if } i = 2l(\lambda), \\ 0 & \text{if otherwise.} \end{cases}$$

Hence, we obtain

$$\overline{H}_i(\mathfrak{w}\mathrm{Gr}(d, n)) \cong \begin{cases} \bigoplus_{\dim \mathfrak{w}\mathrm{Gr}(d, n) - 2l(\lambda) = i} \mathbb{Q} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Since  $\mathfrak{w}\mathrm{Gr}(d, n)$  is covered by the locally contractible charts  $\{\mathfrak{w}U^\lambda\}$ , we see that  $\mathfrak{w}\mathrm{Gr}(d, n)$  is a compact locally contractible space. Hence, the singular homology and the Borel-Moore homology agree ([23, Lem.14, sec.10, chap.6]):

$$H_i(\mathfrak{w}\mathrm{Gr}(d, n)) \cong \overline{H}_i(\mathfrak{w}\mathrm{Gr}(d, n)).$$

By applying the rational Poincaré duality (c.f. [1, Proposition 1.28]), we obtain the claim.  $\square$

Recall that the rational equivariant cohomology for the  $\mathfrak{w}R$ -action on  $\mathfrak{w}\mathrm{Gr}(d, n)$  is defined as the cohomology of the *Borel construction*, i.e. the total space of the fibration

$$\mathfrak{w}\mathrm{Gr}(d, n) \xrightarrow{\zeta} E\mathfrak{w}R \times_{\mathfrak{w}R} \mathfrak{w}\mathrm{Gr}(d, n) \rightarrow B\mathfrak{w}R,$$

where  $EwR \rightarrow BwR$  is a universal principal  $wR$ -bundle with the contractible total space and  $EwR \times_{wR} wGr(d, n) := (EwR \times wGr(d, n))/wR$ . The pullback of the projection to  $BwR$  defines the  $H^*(BwR)$ -module structure of  $H_{wR}^*(wGr(d, n))$ . Since the fiber  $wGr(d, n)$  is path-connected, the vanishing of odd degree classes implies that the Serre spectral sequence of this fibration collapses at  $E_2$ -stage. This implies the freeness of  $H_{wR}^*(wGr(d, n))$  as a  $H^*(BwR)$ -module:

**Proposition 2.13.** *As  $H^*(BwR)$ -modules,*

$$H_{wR}^*(wGr(d, n)) \cong H^*(BwR) \otimes_{\mathbb{Q}} H^*(wGr(d, n)).$$

*In particular,  $H_{wR}^*(wGr(d, n))$  is a free module over  $H^*(BwR)$ .*

### 3. Equivariant Weighted Schubert Classes

Recall that  $T, wR$  and  $R$  be the real tori in  $T_{\mathbb{C}}, wR_{\mathbb{C}}, R_{\mathbb{C}}$  respectively. In this section, we discuss the relations among the rational equivariant cohomologies  $H_T^*(aPl(d, n)^\times)$ ,  $H_{wR}^*(wGr(d, n))$ , and  $H_R^*(Gr(d, n))$ . In fact, they are isomorphic as rings, while they are modules over different polynomial rings. In  $H_T^*(aPl(d, n)^\times)$ , there are geometrically defined cohomology classes  $a\tilde{S}_\lambda$  associated to the varieties  $a\Omega_\lambda$ . We define our *equivariant weighted Schubert classes*  $w\tilde{S}_\lambda$  in  $H_{wR}^*(wGr(d, n))$  as the classes corresponding to  $a\tilde{S}_\lambda$  under the isomorphism.

The quotient maps from  $aPl(d, n)^\times$  to  $wGr(d, n)$  and  $Gr(d, n)$ , and from  $T$  to  $wR$  and  $R$ , induce the following commutative diagram of the Borel constructions:

$$\begin{array}{ccccc} ER \times_R Gr(d, n) & \xleftarrow{h} & ET \times_T aPl(d, n)^\times & \xrightarrow{wh} & EwR \times_{wR} wGr(d, n) \\ \downarrow & & \downarrow & & \downarrow \\ BR & \xleftarrow{\quad} & BT & \xrightarrow{\quad} & BwR \end{array}$$

By the functoriality, the pullback maps

$$h^* : H_R^*(Gr(d, n)) \rightarrow H_T^*(aPl(d, n)^\times) \quad \text{and} \quad wh^* : H_{wR}^*(wGr(d, n)) \rightarrow H_T^*(aPl(d, n)^\times)$$

are homomorphism of rings over the polynomial rings  $H^*(BR)$  and  $H^*(BwR)$ . The proof of the following proposition is postponed until after we define the weighted Schubert classes.

**Proposition 3.1.** *The maps  $h^*$  and  $wh^*$  are isomorphisms as rings over the polynomial rings  $H^*(BR)$  and  $H^*(BwR)$  respectively.*

**Definition 3.2.** Let  $\lambda \in \{d\}$ . Since each variety  $a\Omega_\lambda$  is a closed  $T$ -invariant irreducible subvariety in a non-singular quasi-projective  $T$ -variety  $aPl(d, n)^\times$ , the standard argument (c.f. [8, Appendix B]) allows us to define the  $T$ -equivariant class  $[a\Omega_\lambda]_T$  associated to  $a\Omega_\lambda$  in  $H_T^*(aPl(d, n)^\times)$ :

$$a\tilde{S}_\lambda := [a\Omega_\lambda]_T \in H_T^{2l(\lambda)}(aPl(d, n)^\times).$$

We define the  $wR$ -equivariant weighted Schubert class by

$$w\tilde{S}_\lambda := (wh^*)^{-1}(a\tilde{S}_\lambda) \in H_{wR}^{2l(\lambda)}(wGr(d, n)).$$

The equivariant Schubert class  $\tilde{S}_\lambda$  for the ordinary Grassmannian  $Gr(d, n)$  is defined as the  $R$ -equivariant class associated to the ordinary Schubert variety  $\Omega_\lambda$  as usual. We will see that  $\tilde{S}_\lambda$  maps to  $a\tilde{S}_\lambda$  by  $h^*$  in Section 4.6.

**Remark 3.3.** Following [7] and [18],  $H_T^*(\mathrm{aPl}(d, n)^\times)$  can be identified with the  $wR$ -equivariant cohomology of the quotient stack  $[\mathrm{aPl}(d, n)^\times/wD_\mathbb{C}]$  and the isomorphism  $wh^*$  is nothing but the identification of the (equivariant) rational cohomology rings of the *weighted Grassmannian orbifold stack*  $[\mathrm{aPl}(d, n)^\times/wD_\mathbb{C}]$  and its coarse moduli space  $w\mathrm{Gr}(d, n)$ . In these identifications,  $a\tilde{S}_\lambda$  and  $w\tilde{S}_\lambda$  should be regarded as the class associated to the  $wR$ -invariant substack  $[a\Omega_\lambda/wD_\mathbb{C}]$ . It should then coincide with the Poincaré dual of the cycle  $[w\Omega_\lambda]$  up to the multiplicity of the substack  $[a\Omega_\lambda/wD_\mathbb{C}]$  in  $[\mathrm{aPl}(d, n)^\times/wD_\mathbb{C}]$ . We leave this aspect of the theory to elsewhere.

**Proof of Proposition 3.1.** The claim follows essentially from the *Vietoris-Begle mapping theorem*, but we need to prepare the description of  $w\mathrm{Gr}(d, n)$  as the quotient of a compact space by a real torus.

Since the  $wD$ -action on  $\mathbb{C}\{\frac{n}{d}\}$  factors through the canonical  $(S^1)\{\frac{n}{d}\}$ -action, it is hamiltonian with the standard moment map. Since  $\mathrm{aPl}(d, n)^\times$  is a  $wD$ -invariant symplectic submanifold of  $\mathbb{C}\{\frac{n}{d}\}$ , there is the induced moment map<sup>1</sup>

$$\Psi : \mathrm{aPl}(d, n)^\times \rightarrow \mathbb{R} \quad ; \quad x \mapsto -\frac{1}{2} \sum_{\lambda \in \{\frac{n}{d}\}} dw_\lambda |x_\lambda|^2.$$

For a regular value  $\xi$ , the preimage  $M := \Psi^{-1}(\xi)$  is a compact  $T$ -invariant submanifold of  $\mathrm{aPl}(d, n)^\times$ . Moreover there is a  $T$ -equivariant deformation retraction from  $\mathrm{aPl}(d, n)^\times$  to  $M$  given by the homotopy

$$F : \mathrm{aPl}(d, n)^\times \times I \rightarrow \mathrm{aPl}(d, n)^\times \quad ; \quad (x, s) \mapsto \left( (s\sqrt{\xi/\Psi(x)} + (1-s))x_\lambda \right)_{\lambda \in \{\frac{n}{d}\}}.$$

Thus, the inclusion  $\iota : M \hookrightarrow \mathrm{aPl}(d, n)^\times$  induces the isomorphism:

$$(3.1) \quad \iota^* : H_T^*(M) \longrightarrow H_T^*(\mathrm{aPl}(d, n)^\times).$$

Passing to the quotients, we obtain the  $wR$ -equivariant map  $\bar{\tau} : M/wD \rightarrow w\mathrm{Gr}(d, n)$ . This map can be shown to be a homeomorphism by a direct computation (See also [14, Theorem 7.4]). Hence, we obtain the isomorphism:

$$(3.2) \quad \bar{\tau}^* : H_{wR}^*(M/wD) \longrightarrow H_{wR}^*(w\mathrm{Gr}(d, n)).$$

Let  $\theta : ET \times_T M \rightarrow EwR \times_{wR} M/wD$  be a map induced by the quotient maps  $M \rightarrow M/wD$  and  $T \rightarrow wR$ . Then we have the following commutative diagram.

$$(3.3) \quad \begin{array}{ccc} H_{wR}^*(w\mathrm{Gr}(d, n)) & \xrightarrow{wh^*} & H_T^*(\mathrm{aPl}(d, n)^\times) \\ \cong \downarrow \bar{\tau}^* & & \cong \downarrow \iota^* \\ H_{wR}^*(M/wD) & \xrightarrow{\theta^*} & H_T^*(M) \end{array}$$

Thus  $wh^*$  is an isomorphism if  $\theta^*$  is an isomorphism, which we prove in the next lemma.  $\square$

**Lemma 3.4.** *Let  $M$  be a compact manifold with a smooth action of a torus  $T$ . Let  $G \subset T$  be a subtorus that acts on  $M$  with finite stabilizers. Let  $R := T/G$ . Then the natural map  $\theta : ET \times_T M \rightarrow ER \times_R (M/G)$  induces an isomorphism of rings over  $H^*(BR)$  on the rational equivariant cohomology*

$$\theta^* : H_R^*(M/G) \rightarrow H_T^*(M).$$

<sup>1</sup>Here we identify  $\mathrm{Lie}(wD) \cong \mathrm{Lie}(S^1) = \mathbb{R}$  by the map  $S^1 \rightarrow wD(t \mapsto (t^{dw_1+a}, \dots, t^{dw_n+a}))$ .

*Proof.* We will show that  $\theta^* : H_R^i(M/G) \rightarrow H_T^i(M)$  is a ring isomorphism for each  $i > 0$ . We take  $ET = (S^\infty)^n$  and  $BT = (\mathbb{CP}^\infty)^n$  where  $\dim T = n$ . The inclusions  $ET^m := (S^{2m+1})^n \rightarrow ET$  and  $BT^m := (\mathbb{CP}^m)^n \rightarrow BT$  approximate the bundle  $ET \rightarrow BT$  and induce the canonical isomorphism  $a^* : H_T^i(M) \rightarrow H^i(ET^m \times_T M)$  for a sufficiently large  $m$ . Consider the composition  $\varphi : ET^m \hookrightarrow ET \rightarrow ER$  of the approximation map and a universal map for the quotient map  $T \rightarrow R$ . There is a commutative diagram and its pullback diagram:

$$\begin{array}{ccc}
 ER \times_R (M/G) & \xleftarrow{\theta} & ET \times_T M \\
 & \searrow \theta_m & \uparrow a \\
 & & ET^m \times_T M
 \end{array}
 \qquad
 \begin{array}{ccc}
 H_R^i(M/G) & \xrightarrow{\theta^*} & H_T^i(M) \\
 & \searrow \theta_m^* & \downarrow a^* \\
 & & H^i(ET^m \times_T M).
 \end{array}$$

Since  $a^*$  is an isomorphism for large  $m$ , it suffices to show that  $\theta_m^*$  is an isomorphism. Observe that we can write  $\theta_m = f \circ s$  where

$$\begin{aligned}
 s : ET^m \times_T M &\rightarrow ER \times_R (ET^m \times_G M) \quad ; \quad [\alpha, x]_T \mapsto [\varphi(\alpha), [\alpha, x]_G]_R, \\
 f : ER \times_R (ET^m \times_G M) &\rightarrow ER \times_R (M/G) \quad ; \quad [\beta, [\alpha, x]_G]_R \mapsto [\beta, [x]_G]_R.
 \end{aligned}$$

It is easy to see that  $s$  is a section of  $g : ER \times_R (ET^m \times_G M) \rightarrow ET^m \times_T M$  sending  $[\beta, [\alpha, x]_G]_R \mapsto [\alpha, x]_T$  whose fiber is the contractible space  $ER$ . Therefore  $s^*$  is an isomorphism<sup>2</sup>. On the other hand, the preimage of  $f$  at  $[\beta, [x]_G]_R$  is homeomorphic to  $ET^m/G_x$  where  $G_x$  is the isotropy of the  $G$ -action at  $x \in M$ . Let us denote  $\bar{H}^*$  the rational Alexander-Spanier cohomology. Then we have<sup>3</sup>

$$(3.4) \quad \bar{H}^p(ET^m/G_x) = 0 \quad \text{for } 0 < p < m.$$

The projection  $f' : ET^m \times_G M \rightarrow M/G$  is closed since it is a map from a compact space to a Hausdorff space. This implies that  $f$  is a closed surjection, by chasing the following commutative diagram of projections

$$\begin{array}{ccc}
 ER \times (ET^m \times_G M) & \xrightarrow{(id, f')} & ER \times (M/G) \\
 \downarrow & & \downarrow \\
 ER \times_R (ET^m \times_G M) & \xrightarrow{f} & ER \times_R (M/G)
 \end{array}$$

where the vertical maps are closed since  $R$  is compact (c.f. Proposition 1.58 in [12]). Observe that  $ER \times_R (ET^m \times_G M)$  and  $ER \times_R (M/G)$  are locally contractible, paracompact Hausdorff spaces<sup>4</sup>. Together with (3.4), the Vietoris-Begle mapping theorem ([23, Thm.15, Sec.9, Chp.6]) shows that

$$f^* : \bar{H}^p(ER \times_R (ET^m \times_G M)) \rightarrow \bar{H}^p(ER \times_R (M/G))$$

is an isomorphism for all  $p < m$ . Since the Alexander-Spanier cohomology and the singular cohomology are naturally isomorphic for locally contractible paracompact Hausdorff spaces ([23, Cor.5, Sec.9, Chp.6]), the pullback  $f^*$  for the singular cohomologies is also an isomorphism for

<sup>2</sup>This follows from [6]; Corollary 3.2. Note that every open covering of the base of the  $EwR$ -bundle  $g$  is numerable since the base of  $g$  is compact (see [6]; p.226).

<sup>3</sup>We can use Theorem 5.30 in [12] with Theorem 5.8 and the comment in Example 1 in p. 250 since  $\bar{H}^*$  and  $H^*$  are isomorphic for locally contractible paracompact Hausdorff spaces. Observe that the  $G_x$ -action comes from the  $T$ -action.

<sup>4</sup> $M/G$  is locally contractible because of the equivariant tubular neighborhood theorem.

all degree  $p < m$ . Finally we can conclude that  $\theta_m^*$  is an isomorphism for all degree  $p < m$  since  $\theta_m = f \circ s$ .  $\square$

#### 4. GKM Descriptions and Schubert Classes

In this section, we study the combinatorial presentations of  $H_{\text{wR}}^*(\text{wGr}(d, n))$  and  $H_T^*(\text{aPl}(d, n)^\times)$ , now known as the *GKM theory* developed in [9]. This allows us, in particular, to show that the equivariant weighted Schubert classes  $\text{w}\tilde{S}_{\lambda, \lambda \in \{ \binom{n}{d} \}}$  form an  $H^*(B\text{wR})$ -module basis of  $H_{\text{wR}}^*(\text{wGr}(d, n))$ .

Recall that  $H^*(BT)$  can be canonically identified with the symmetric algebra  $\text{Sym}(\text{Lie}(T)_{\mathbb{Z}}^* \otimes \mathbb{Q})$  where  $\text{Lie}(T)_{\mathbb{Z}}^*$  is the space of  $\mathbb{Z}$ -linear functions on the integral lattice  $\text{Lie}(T)_{\mathbb{Z}} \subset \text{Lie}(T)$ . Since  $T = (S^1)^n$ , we identify its Lie algebra  $\text{Lie}(T)$  with  $\mathbb{R}^n$ , so that we have the  $\mathbb{Z}$ -basis  $\{y_1, \dots, y_n\}$  of  $\text{Lie}(T)_{\mathbb{Z}}^*$  dual to the standard basis of  $\text{Lie}(T)_{\mathbb{Z}}$ . Let

$$\mathbb{Q}[T^*] := H^*(BT) = \text{Sym}(\text{Lie}(T)_{\mathbb{Z}}^* \otimes \mathbb{Q}) = \mathbb{Q}[y_1, \dots, y_n].$$

We adapt the same notation for all other tori except that  $T$  is the only standard torus such that the canonical generators  $y_i$ 's of  $H^*(BT)$  are given. The quotient map  $T \rightarrow \text{wR}$  induces the injection  $\text{Lie}(\text{wR})_{\mathbb{Z}}^* \hookrightarrow \text{Lie}(T)_{\mathbb{Z}}^*$  and hence we will identify  $\mathbb{Q}[\text{wR}^*]$  with the image of the induced embedding  $\mathbb{Q}[\text{wR}^*] \hookrightarrow \mathbb{Q}[T^*]$ . Similarly we will identify  $\mathbb{Q}[R^*]$  with the image of  $\mathbb{Q}[R^*] \hookrightarrow \mathbb{Q}[T^*]$ .

We shall start with the GKM theory for  $R$ -action on  $\text{Gr}(d, n)$  studied in [16]. The  $R$ -fixed points in  $\text{Gr}(d, n)$  are the points  $[e_\mu]$  corresponding to the vector  $e_\mu$  in  $\text{aPl}(d, n)^\times$ . We identify  $H_R^*([e_\mu])$  with  $\mathbb{Q}[R^*]$ . The restriction map to the fixed points

$$(4.1) \quad H_R^*(\text{Gr}(d, n)) \rightarrow \bigoplus_{\mu \in \{ \binom{n}{d} \}} \mathbb{Q}[R^*]; \quad \gamma \mapsto (\gamma|_\mu)_{\mu \in \{ \binom{n}{d} \}}$$

is injective and the image is given by

$$(4.2) \quad \left\{ \alpha = (\alpha(\mu))_\mu \in \bigoplus_{\mu \in \{ \binom{n}{d} \}} \mathbb{Q}[R^*] \left| \begin{array}{l} \alpha(\lambda) - \alpha(\mu) \text{ is divisible by } y_\lambda - y_\mu \\ \text{for any } \lambda \text{ and } \mu \text{ such that } |\lambda \cap \mu| = d - 1 \end{array} \right. \right\}.$$

where  $y_\mu := \sum_{i \in \mu} y_i$  for all  $\mu \in \{ \binom{n}{d} \}$ . Note that  $y_\lambda - y_\mu \in \text{Lie}(R)_{\mathbb{Z}}^*$  since the linear function  $y_\lambda - y_\mu$  restricted to  $\text{Lie}(D)_{\mathbb{Z}}$  is 0.

**Remark 4.1.** In [16], the authors consider the action of  $T' = (S^1)^n$  on  $\text{Gr}(d, n)$  though a map  $T' \rightarrow R$ . Their presentation is valid in our set-up since  $H_R^*(\text{Gr}(d, n))$  injects to  $H_{T'}^*(\text{Gr}(d, n))$  as rings over  $\mathbb{Q}[R^*]$ .

Now we turn to  $\text{wGr}(d, n)$  and then  $\text{aPl}(d, n)^\times$ . The fixed points of the  $\text{wR}$ -action on  $\text{wGr}(d, n)$  are also the points  $[e_\mu] \in \text{wGr}(d, n)$  corresponding to the vectors  $e_\mu$ . We can naturally identify  $H_{\text{wR}}^*([e_\mu]) \cong \mathbb{Q}[\text{wR}^*]$  and so we have the restriction map

$$(4.3) \quad H_{\text{wR}}^*(\text{wGr}(d, n)) \longrightarrow \bigoplus_{\mu \in \{ \binom{n}{d} \}} \mathbb{Q}[\text{wR}^*], \quad \gamma \mapsto (\gamma|_\mu)_{\mu \in \{ \binom{n}{d} \}}$$

For  $\text{aPl}(d, n)^\times$ , we restrict  $H_T(\text{aPl}(d, n)^\times)$  to complex 1-dimensional orbits of  $T_{\mathbb{C}}$  instead of restricting to the fixed points. The complex 1-dimensional orbits of  $T_{\mathbb{C}}$  are given by  $\mathbb{C}^\times e_\mu$ . For each  $\mu \in \{ \binom{n}{d} \}$ , let  $T_\mu$  be the isotropy subgroup at  $e_\mu$  for the  $T$ -action on  $\text{aPl}(d, n)^\times$ . It is the kernel of the map  $T \rightarrow S^1$  sending  $(t_1, \dots, t_n)$  to  $t_\mu := t_{\mu_1} \cdots t_{\mu_d}$  so that it is not hard to see

that  $T_\mu$  is connected. Thus, with the natural isomorphisms  $H_T(\mathbb{C}^\times e_\mu) \cong H_{T_\mu}(e_\mu) \cong \mathbb{Q}[T_\mu^*]$ , we obtain

$$(4.4) \quad H_T^*(\mathrm{aPl}(d, n)^\times) \longrightarrow \bigoplus_{\mu \in \{n\}_d} \mathbb{Q}[T_\mu^*], \quad P \mapsto (P|_\mu)_{\mu \in \{n\}_d}$$

Putting (4.1, 4.3, 4.4) together with  $h^*$  and  $wh^*$ , we have the commutative diagram

$$(4.5) \quad \begin{array}{ccc} H_R^*(\mathrm{Gr}(d, n)) & \longrightarrow & \bigoplus_\mu \mathbb{Q}[R^*] \\ h^* \downarrow \cong & & \downarrow \cong \kappa^* \\ H_T^*(\mathrm{aPl}(d, n)^\times) & \longrightarrow & \bigoplus_\mu \mathbb{Q}[T_\mu^*] \\ wh^* \uparrow \cong & & \uparrow \cong w\kappa^* \\ H_{wR}^*(w\mathrm{Gr}(d, n)) & \longrightarrow & \bigoplus_\mu \mathbb{Q}[wR^*] \end{array}$$

where the right vertical maps are induced from  $\kappa_\mu : T_\mu \rightarrow T \rightarrow R$  and  $w\kappa_\mu : T_\mu \rightarrow T \rightarrow wR$  and they are isomorphisms because  $\kappa_\mu$  and  $w\kappa_\mu$  have finite (or trivial) kernels. Also note that the inclusion  $T_\mu \rightarrow T$  induces the surjection  $\mathrm{Lie}(T)_\mathbb{Z}^* \rightarrow \mathrm{Lie}(T_\mu)_\mathbb{Z}^*$  and  $\mathrm{Lie}(T_\mu)_\mathbb{Z}^* \cong \mathrm{Lie}(T)_\mathbb{Z}^*/(y_\mu)$ . Thus we identify

$$\mathbb{Q}[T_\mu^*] \cong \mathbb{Q}[T^*]/(y_\mu).$$

The following are obtained by translating (4.2) to  $H_T^*(\mathrm{aPl}(d, n)^\times)$  and  $H_{wR}^*(w\mathrm{Gr}(d, n))$  via the diagram (4.5).

**Proposition 4.2** (GKM for  $w\mathrm{Gr}(d, n)$ ). *The restriction map (4.3) is injective and the image is given by*

$$\left\{ \alpha \in \bigoplus_{\mu \in \{n\}_d} \mathbb{Q}[wR^*] \left| \begin{array}{l} \alpha(\lambda) - \alpha(\mu) \text{ is divisible by } w_\mu y_\lambda - w_\lambda y_\mu \\ \text{for any } \lambda \text{ and } \mu \text{ such that } |\lambda \cap \mu| = d-1 \end{array} \right. \right\}.$$

Here note that  $w_\mu y_\lambda - w_\lambda y_\mu \in \mathbb{Q}[wR^*]$ .

**Proposition 4.3** (GKM for  $\mathrm{aPl}(d, n)^\times$ ). *The restriction map (4.4) is injective and the image is given by*

$$\left\{ P \in \bigoplus_{\mu \in \{n\}_d} \mathbb{Q}[T_\mu^*] \left| \begin{array}{l} P(\lambda) = P(\mu) \quad \text{in } \mathbb{Q}[T^*]/(y_\lambda, y_\mu) \\ \text{for any } \lambda \text{ and } \mu \text{ such that } |\lambda \cap \mu| = d-1 \end{array} \right. \right\}.$$

*Proof of Proposition 4.3 and Proposition 4.2*

The injectivity of the maps (4.3) and (4.4) follows from the injectivity of the map (4.1) by the commutativity of the diagram (4.5). What is left is to check that the GKM conditions are equivalent under the isomorphisms  $\kappa^*$  and  $w\kappa^*$ . We prove it for  $w\kappa$  because  $\kappa$  is a special case of  $w\kappa$ . First note that, in Proposition 4.2,  $\alpha(\lambda) - \alpha(\mu)$  is divisible by  $w_\mu y_\lambda - w_\lambda y_\mu$  if and only if  $\alpha(\lambda) - \alpha(\mu) = 0$  in  $\mathbb{Q}[wR^*]/(w_\mu y_\lambda - w_\lambda y_\mu)$ . Therefore the GKM conditions are equivalent under  $w\kappa^*$  if  $w\kappa_\lambda^*$  and  $w\kappa_\mu^*$  induce the isomorphism

$$\frac{\mathbb{Q}[wR^*]}{(w_\mu y_\lambda - w_\lambda y_\mu)} \rightarrow \frac{\mathbb{Q}[T^*]}{(y_\lambda, y_\mu)}, \quad f \mapsto w\kappa_\lambda^*(f) = w\kappa_\mu^*(f).$$

This map is obviously well-defined. It is also easy to check that this is an isomorphism.  $\square$

**Remark 4.4.** Proposition 4.2 can be proved directly as a consequence of [9, Theorem 7.2] by studying the data of 0 and 1-dimensional  $wR$ -orbits and Lie algebras of isotropic subgroups of  $wR$ -action on  $wGr(d, n)$ . For this alternative proof, see Section 7.3 in the appendix. Proposition 4.3 can be also shown directly from Theorem 5.5 in [20] by using the description of  $wGr(d, n)$  as the symplectic quotient of  $aPl(d, n)^\times$  by the real torus  $wD$  explained in Section 3.

In the rest of the section, we compute the certain values of the Schubert classes  $a\tilde{S}_\lambda$  and  $w\tilde{S}_\lambda$  under the restriction maps. As a corollary, we show that the Schubert classes  $\tilde{S}_\lambda$  in  $H_R^*(Gr(d, n))$  correspond to the classes  $a\tilde{S}_\lambda$  in  $H_T^*(aPl(d, n)^\times)$  under  $h^*$  as expected. Also we show that the weighted Schubert classes  $w\tilde{S}_\lambda$  will form an  $H^*(BwR)$ -module basis of  $H_{wR}^*(wGr(d, n))$ .

**Proposition 4.5.**

$$a\tilde{S}_\lambda|_\mu = \begin{cases} 0 & \text{if } \mu \not\geq \lambda, \\ \prod_{(k,l) \in \text{inv}(\lambda)} y_{(k,l)\lambda} & \text{if } \mu = \lambda. \end{cases} \quad \text{in } \mathbb{Q}[T^*]/(y_\mu)$$

*Proof.* Let  $Y := aPl(d, n)^\times$  for brevity. By the construction (c.f. [8, Appedix B.3]), the class  $a\tilde{S}_\lambda$  maps to the equivariant Euler class  $\chi_T(N^\circ)$  by the pullback along the inclusion  $a\Omega_\lambda^\circ \hookrightarrow Y$  where  $N^\circ$  is the normal bundle of  $a\Omega_\lambda^\circ$  in  $Y$ . This maps further to  $\chi_{T_\lambda}(N_{e_\lambda}^\circ)$  via the pullback map  $H_T^*(a\Omega_\lambda^\circ) \rightarrow H_{T_\mu}^*(e_\mu)$  where  $N_{e_\mu}^\circ$  is the fiber of  $N^\circ$  at  $e_\mu$ . Since the normal bundle  $N^\circ$  is  $T$ -equivariantly identified with  $aU^\lambda$ , the equivariant chart  $\psi_\lambda$  given at (2.5) allows us to find the weight of the representation  $T_\lambda \curvearrowright N_{e_\lambda}$  to be

$$(4.6) \quad \prod_{(k,l) \in \text{inv}(\lambda)} y_{(k,l)\lambda} \quad \text{as an element of } \text{Lie}(T)_\mathbb{Z}^*/(y_\lambda) = \text{Lie}(T_\lambda)_\mathbb{Z}^*.$$

This proves the case when  $\mu = \lambda$ . The case  $\mu \not\geq \lambda$  follows from  $e_\mu \notin a\Omega_\lambda$  (Proposition 2.7).  $\square$

It is a well-known fact that  $\tilde{S}_\lambda|_\lambda = \prod_{(k,l) \in \text{inv}(\lambda)} (y_{(k,l)\lambda} - y_\lambda)$  and  $\tilde{S}_\lambda|_\lambda = 0$  for all  $\mu \not\geq \lambda$  (c.f. [16]). Also a class having such values at fixed points is unique [16, LEMMA 1]. Therefore Proposition 4.5, together with the diagram 4.5, has the following immediate corollary.

**Corollary 4.6.** *For each  $\lambda \in \{d\}$ ,  $h^*(\tilde{S}_\lambda) = a\tilde{S}_\lambda$ .*

The next proposition is also immediate from Proposition 4.5.

**Proposition 4.7.**

$$w\tilde{S}_\lambda|_\mu = \begin{cases} 0 & \text{if } \mu \not\geq \lambda, \\ \prod_{(k,l) \in \text{inv}(\lambda)} \left( y_{(k,l)\lambda} - \frac{w_{(k,l)\lambda}}{w_\lambda} y_\lambda \right) & \text{if } \mu = \lambda. \end{cases}$$

*Proof.* Since  $w\tilde{S}_\lambda := (wh^*)^{-1}(a\tilde{S}_\lambda)$  and  $y_{(k,l)\lambda} - \frac{w_{(k,l)\lambda}}{w_\lambda} y_\lambda = y_{(k,l)\lambda}$  in  $\mathbb{Q}[T^*]/\langle y_\lambda \rangle$ , we only need to check that

$$(4.7) \quad y_{(k,l)\lambda} - \frac{w_{(k,l)\lambda}}{w_\lambda} y_\lambda \in \text{Lie}(wR)_\mathbb{Z}^* \otimes \mathbb{Q}.$$

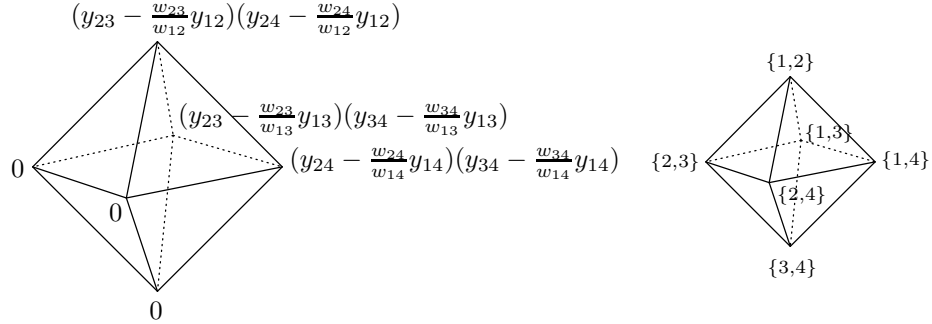
This can be checked by a straightforward calculation.  $\square$

Having the upper-triangularity of the weighted Schubert classes as above, the proof of [16, Proposition 1] can be applied words by words to obtain

**Proposition 4.8.**  *$\{w\tilde{S}_\lambda\}_\lambda$  is an  $H^*(BwR)$ -module basis of  $H_{wR}^*(wGr(d, n))$ .*



**Example 4.9.** The followings is  $w\tilde{S}_{24}$  in  $H_{wR}^*(wGr(2, 4))$ :



where the vertices are the elements of  $\{\frac{4}{2}\}$  and there is an edge for each pair of  $\lambda$  and  $\mu$  satisfying  $|\lambda \cap \mu| = 1$ .

## 5. Structure Constants and Positivity

Since  $\{w\tilde{S}_\lambda\}_\lambda$  is an  $H^*(BwR)$ -module basis of  $H_{wR}^*(wGr(d, n))$ , we can expand their pairwise cup product uniquely over  $H^*(BwR)$ :

$$(5.1) \quad w\tilde{S}_\lambda w\tilde{S}_\mu = \sum_{\nu} w\tilde{c}_{\lambda\mu}^\nu w\tilde{S}_\nu \quad \text{where } w\tilde{c}_{\lambda\mu}^\nu \in H^*(BwR).$$

Knutson-Tao [16] gave an explicit combinatorial formula for the equivariant structure constants  $\tilde{c}_{\lambda\mu}^\nu$  of the  $R$ -equivariant cohomology of  $Gr(d, n)$  in terms of the equivariant puzzles. Their formula of  $\tilde{c}_{\lambda\mu}^\nu$  is *manifestly positive* in a sense that  $\tilde{c}_{\lambda\mu}^\nu$  is a polynomial in  $u_i$ 's with non-negative coefficients where  $\{u_i := y_{i+1} - y_i\}$  is a basis of  $Lie(R)_\mathbb{Z}^*$ . In this section, we derive the formula for  $w\tilde{c}_{\lambda\mu}^\nu$  from their formula by passing it through  $H_T(aPl(d, n)^\times)$  via  $h^*$  and  $wh^*$ . Also we find a  $\mathbb{Q}$ -basis  $\{wu_i\}$  of  $Lie(wR)_\mathbb{Z}^* \otimes \mathbb{Q}$  such that our formula of  $w\tilde{c}_{\lambda\mu}^\nu$  is manifestly positive with respect to the basis  $\{wu_i\}$  when  $w_1 \leq \dots \leq w_n$ . The manifestly positive formula for the structure constants  $\{w\tilde{c}_{\lambda\mu}^\nu\}$  of the ordinary cohomology  $H^*(wGr(d, n))$  is also obtained by specializing the one for  $w\tilde{c}_{\lambda\mu}^\nu$  at  $wu_1 = \dots = wu_{n-1} = 0$ .

We start with introducing new terminologies which extend the ones provided in [16, p.227]. For every puzzle, we choose a total order on the set of equivariant pieces once and for all. Let  $P$  be a puzzle satisfying  $\partial P = \Delta_{\lambda\mu}^\nu$ . Let  $p$  be an equivariant piece in  $P$ , whose weight is  $wt(p) = y_j - y_i$  where  $j > i$ , i.e.  $p$  pokes out the  $i$ -th and  $j$ -th place on the south side of  $P$ . For each  $\xi \in \{\frac{n}{d}\}$ , we define the  $w$ -weight of  $p$  with respect to  $\xi$  by

$$(5.2) \quad wt^\xi(p) := (y_j - y_i) - \frac{w(p)}{w_\xi} y_\xi \in \mathbb{Q}[wR^*] \quad \text{where } w(p) := w_j - w_i.$$

Let  $(p_1, \dots, p_r)$  be the ordered set of equivariant pieces in  $P$ . For each covering sequence  $\xi^k \rightarrow \dots \rightarrow \xi^0$  in  $\{\frac{n}{d}\}$  with  $k \leq r$ , we define the  $w$ -weight of  $P$  to be an element of  $\mathbb{Q}[wR^*]$

$$(5.3) \quad wt^{\xi^0, \dots, \xi^k}(P) := \sum_{1 \leq i_1 < \dots < i_k \leq r} \frac{w(p_{i_1})}{w_{\xi^0}} \dots \frac{w(p_{i_k})}{w_{\xi^{k-1}}} \frac{\prod_{i=1}^r wt^{\xi^{s(i)}}(p_i)}{wt^{\xi^1}(p_{i_1}) \dots wt^{\xi^k}(p_{i_k})}.$$

where  $s$  is a function on  $\{1, \dots, r\}$  defined by

$$s(i) := \begin{cases} 0 & \text{if } i < i_1 \\ l & \text{if } i_l \leq i < i_{l+1}, l = 1, \dots, k-1 \\ k & \text{if } i_k \leq i \end{cases}$$

As a special case when  $k = 0$ , we have

$$(5.4) \quad \text{wt}^\xi(P) = \text{wt}^\xi(p_1) \cdots \text{wt}^\xi(p_r).$$

Remark that this expression (5.3) depends on the order of the equivariant pieces in  $P$  in general.

**Lemma 5.1.** *Let  $\text{id}$  be the unique minimum in  $\{^n_d\}$  with respect to the Bruhat order and  $\text{div}$  the unique element with  $l(\text{id}) = 1$ . Let  $\nu \in \{^n_d\}$ . Then*

$$(5.5) \quad \mathbf{a}\tilde{S}_{\text{div}} = y_{\text{id}} ;$$

$$(5.6) \quad 0 = -y_\nu \mathbf{a}\tilde{S}_\nu + \sum_{\nu' \rightarrow \nu} \mathbf{a}\tilde{S}_{\nu'} .$$

*Proof.* Since  $\tilde{S}_{\text{div}}|_\mu = y_{\text{id}} - y_\mu$  ([16, Lemma 3]) for each  $\mu \in \{^n_d\}$ , we have

$$\mathbf{a}\tilde{S}_{\text{div}}|_\mu = y_{\text{id}} - y_\mu = y_{\text{id}}, \quad \text{in } \mathbb{Q}[T^*]/(y_\mu).$$

Therefore (5.5) holds:  $\mathbf{a}\tilde{S}_{\text{div}} = y_{\text{id}} \mathbf{a}\tilde{S}_{\text{id}} = y_{\text{id}} \cdot 1$ . On the other hand, the equivariant Pieri-rule given in [16, Proposition 2] holds also in  $H_T^*(\mathbf{aPl}(d, n)^\times)$  by the isomorphism  $h^*$ , and hence we have

$$(5.7) \quad \mathbf{a}\tilde{S}_{\text{div}} \mathbf{a}\tilde{S}_\nu = (y_{\text{id}} - y_\nu) \mathbf{a}\tilde{S}_\nu + \sum_{\nu' \rightarrow \nu} \mathbf{a}\tilde{S}_{\nu'}$$

which, together with (5.5), implies (5.6).  $\square$

The following is the essential equation, immediate from (5.6), to relate the  $\mathbb{Q}[R^*]$ -action to the  $\mathbb{Q}[wR^*]$ -action in  $H_T^*(\mathbf{aPl}(d, n)^\times)$ .

**Proposition 5.2.** *Let  $p$  be an equivariant piece of a puzzle. Then*

$$\text{wt}(p) \mathbf{a}\tilde{S}_\nu = \text{wt}^\nu(p) \mathbf{a}\tilde{S}_\nu + \sum_{\nu' \rightarrow \nu} \frac{w(p)}{w_\nu} \mathbf{a}\tilde{S}_{\nu'}, \quad \text{in } H_T^*(\mathbf{aPl}(d, n)^\times).$$

*Proof.* Let  $\text{wt}(p) = y_j - y_i$  be the weight of  $p$ . The formula is obtained by multiplying  $(w_j - w_i)/w_\nu$  and then adding  $(y_j - y_i) \mathbf{a}\tilde{S}_\nu$  to the equation (5.6).  $\square$

Now we are ready to prove the main theorem of this section.

**Theorem 5.3.** *For each  $\lambda, \mu, \nu \in \{^n_d\}$ , the equivariant structure constant  $w\tilde{c}_{\lambda\mu}^\nu$  is given by*

$$(5.8) \quad w\tilde{c}_{\lambda\mu}^\nu = \left( \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\nu}} \text{wt}^\nu(P) \right) + \sum_{\substack{\nu \rightarrow \nu^1 \rightarrow \\ \dots \rightarrow \nu^k \geq \lambda, \mu}} \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{\lambda\mu}^{\nu^k}}} \text{wt}^{\nu^k, \dots, \nu^1, \nu}(Q).$$

Remark that the  $w$ -weights in this expression depend on the orders of the equivariant pieces in  $P$  and  $Q$ , but  $w\tilde{c}_{\lambda\mu}^\nu$  doesn't depend on the order.

*Proof.* By the isomorphism  $h^*$ , Theorem 2 in [16] translates to

$$\mathbf{a}\tilde{S}_\lambda \mathbf{a}\tilde{S}_\mu = \sum_{\eta \geq \lambda, \mu} \overbrace{\sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\eta}} \text{wt}(p_r) \cdots \text{wt}(p_1) \mathbf{a}\tilde{S}_\eta}^{\tilde{c}_{\lambda\mu}^\eta} \quad \text{in } H_T^*(\mathbf{aPl}(d, n)^\times),$$

where  $(p_1, \dots, p_r)$  denotes an ordered set of all equivariant pieces in  $P$ . Remark that the number of equivariant pieces in  $P$  must be  $l(\lambda) + l(\mu) - l(\eta)$ . For each  $l \leq r$ , by applying Proposition 5.2 repeatedly, we obtain

$$\begin{aligned} \text{wt}(p_l) \cdots \text{wt}(p_1) \mathbf{a}\tilde{S}_\eta &= \text{wt}^\eta(p_1) \cdots \text{wt}^\eta(p_l) \mathbf{a}\tilde{S}_\eta + \sum_{k=1}^l \sum_{\substack{\eta^k \rightarrow \dots \\ \rightarrow \eta^1 \rightarrow \eta}} \left( \sum_{1 \leq i_1 < \dots < i_k \leq l} \right. \\ &\quad \text{wt}^\eta(p_1) \cdots \text{wt}^\eta(p_{i_1-1}) \cdot \frac{w(p_{i_1})}{w_\eta} \cdot \text{wt}^{\eta^1}(p_{i_1+1}) \cdots \text{wt}^{\eta^1}(p_{i_2-1}) \cdot \frac{w(p_{i_2})}{w_{\eta^1}} \cdot \text{wt}^{\eta^2}(p_{i_2+1}) \\ &\quad \left. \cdots \text{wt}^{\eta^{k-1}}(p_{i_{k-1}}) \cdot \frac{w(p_{i_k})}{w_{\eta^{k-1}}} \cdot \text{wt}^{\eta^k}(p_{i_k+1}) \cdots \text{wt}^{\eta^k}(p_l) \right) \mathbf{a}\tilde{S}_{\eta^k}. \end{aligned}$$

It is straightforward to prove this formula by an induction on  $l \leq r$ . When  $l = r$ ,

$$\text{wt}(p_r) \cdots \text{wt}(p_1) \mathbf{a}\tilde{S}_\eta = \text{wt}^\eta(P) \mathbf{a}\tilde{S}_\eta + \sum_{k=1}^{l(\lambda)+l(\mu)-l(\eta)} \sum_{\substack{\eta^k \rightarrow \dots \\ \rightarrow \eta^1 \rightarrow \eta}} \text{wt}^{\eta, \eta^1, \dots, \eta^k}(P) \cdot \mathbf{a}\tilde{S}_{\eta^k}.$$

Therefore

$$\begin{aligned} \mathbf{a}\tilde{S}_\lambda \mathbf{a}\tilde{S}_\mu &= \sum_{\eta \geq \lambda, \mu} \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\eta}} \left( \text{wt}^\eta(P) \cdot \mathbf{a}\tilde{S}_\eta + \sum_{k=1}^{l(\lambda)+l(\mu)-l(\eta)} \sum_{\substack{\eta^k \rightarrow \dots \\ \rightarrow \eta^1 \rightarrow \eta}} \text{wt}^{\eta, \eta^1, \dots, \eta^k}(P) \cdot \mathbf{a}\tilde{S}_{\eta^k} \right) \\ &= \sum_{\eta \geq \lambda, \mu} \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\eta}} \text{wt}^\eta(P) \cdot \mathbf{a}\tilde{S}_\eta + \sum_{\substack{\nu \rightarrow \eta^{k-1} \rightarrow \dots \\ \rightarrow \eta^1 \rightarrow \eta \geq \lambda, \mu}} \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\nu}} \text{wt}^{\eta, \eta^1, \dots, \eta^{k-1}, \nu}(P) \cdot \mathbf{a}\tilde{S}_\nu \\ &= \sum_{\nu \geq \lambda, \mu} \left( \left( \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\nu}} \text{wt}^\nu(P) \right) + \sum_{\substack{\nu \rightarrow \nu^1 \rightarrow \dots \\ \dots \rightarrow \nu^k \geq \lambda, \mu}} \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{\lambda\mu}^{\nu^k}}} \text{wt}^{\nu^k, \dots, \nu^1, \nu}(Q) \right) \cdot \mathbf{a}\tilde{S}_\nu. \end{aligned}$$

Since  $\text{wt}^\nu(P)$  and  $\text{wt}^{\nu^k, \dots, \nu^1, \nu}(Q)$  are in  $\mathbb{Q}[\mathbf{wR}^*]$ ,  $wh^*$  maps this equation to the desired equation in  $H_{\mathbf{wR}}^*(\mathbf{wGr}(d, n))$ .  $\square$

Let

$$(5.9) \quad \mathbf{w}u_i := (y_{i+1} - y_i) - \frac{w_{i+1} - w_i}{w_{\text{id}}} y_{\text{id}} \in \mathbb{Q}[\mathbf{wR}^*].$$

We can easily check that  $\{\mathbf{w}u_1, \dots, \mathbf{w}u_{n-1}\}$  is a  $\mathbb{Q}$ -basis of  $\text{Lie}(\mathbf{wR})_{\mathbb{Z}}^* \otimes \mathbb{Q}$ . Then the next positivity theorem is a direct consequence of Theorem 5.3 and Proposition 5.5 which is proved right after.

**Theorem 5.4.** *If  $w_1 \leq w_2 \leq \dots \leq w_n$ , then  $\mathbf{w}\tilde{c}_{\lambda\mu}^\nu$  is a polynomial in  $\mathbf{w}u_1, \dots, \mathbf{w}u_{n-1}$  with non-negative coefficients.*

**Proposition 5.5.** *Let  $P$  be a puzzle whose south string is  $\nu$ . Suppose that  $P$  involves an equivariant piece  $p$ . If  $w_1 \leq w_2 \leq \cdots \leq w_n$ , then  $\text{wt}^\nu(p)$  is a linear combination of  $wu_1, \dots, wu_{n-1}$  with non-negative coefficients.*

*Proof.* We prove, by induction on the length  $l(\nu)$ , that the linear polynomial  $(y_j - y_i) - \frac{w_j - w_i}{w_\nu} y_\nu$  is a linear combination of  $wu_1, \dots, wu_{n-1}$  with non-negative coefficients for each  $1 \leq i < j \leq n$  and each  $\nu \in \{d\}^n$ .

If  $l(\nu) = 0$ , the statement is obvious since  $\nu = \text{id}$ . We assume that the claim holds for all  $\nu'$  with  $l(\nu') \leq m - 1$  for some integer  $m$ . Let  $l(\nu) = m$ . Observe

$$\begin{aligned} & (y_j - y_i) - (w_j - w_i) \frac{y_\nu}{w_\nu} \\ &= \sum_{k=i}^{j-1} \left[ \left( (y_{k+1} - y_k) - (w_{k+1} - w_k) \frac{y_{\text{id}}}{w_{\text{id}}} \right) + (w_{k+1} - w_k) \left( \frac{y_{\text{id}}}{w_{\text{id}}} - \frac{y_\nu}{w_\nu} \right) \right]. \end{aligned}$$

We show that  $\frac{y_{\text{id}}}{w_{\text{id}}} - \frac{y_\nu}{w_\nu}$  is written non-negatively. Let  $\nu^1, \dots, \nu^m \in \{d\}^n$  such that  $\nu = \nu^m \rightarrow \cdots \rightarrow \nu^1 \rightarrow \text{id}$ , and write

$$\frac{y_{\text{id}}}{w_{\text{id}}} - \frac{y_\nu}{w_\nu} = \sum_{s=0}^{m-1} \left( \frac{y_{\nu^s}}{w_{\nu^s}} - \frac{y_{\nu^{s+1}}}{w_{\nu^{s+1}}} \right)$$

where  $\nu_0 := \text{id}$ . Since  $\nu^{s+1} \rightarrow \nu^s$ , there exists an integer  $1 \leq a \leq n$  such that  $\nu^{s+1} = (a, a+1)\nu^s$ , and therefore

$$\frac{y_{\nu^s}}{w_{\nu^s}} - \frac{y_{\nu^{s+1}}}{w_{\nu^{s+1}}} = \frac{1}{w_{\nu^{s+1}}} \left( (y_{a+1} - y_a) - (w_{a+1} - w_a) \frac{y_{\nu^s}}{w_{\nu^s}} \right).$$

By the induction hypothesis, the RHS is a linear combination of  $wu_i$ 's with non-negative coefficients, and so is  $\frac{y_{\text{id}}}{w_{\text{id}}} - \frac{y_\nu}{w_\nu}$  and  $(y_j - y_i) - (w_j - w_i) \frac{y_\nu}{w_\nu}$ .  $\square$

As a corollary of Theorem 5.3, we give an explicit formula of the structure constants in  $H^*(\text{wGr}(d, n))$ . For each  $\lambda \in \{d\}^n$ , define

$$\text{w}S_\lambda := \zeta^*(\text{w}\tilde{S}_\lambda) \in H^*(\text{wGr}(d, n))$$

where  $\zeta^* : H_{\text{wR}}^*(\text{wGr}(d, n)) \rightarrow H^*(\text{wGr}(d, n))$  is the surjection mentioned in Section 2.4. Under the natural isomorphism  $H^*(\text{wGr}(d, n)) \cong H_{\text{wD}}^*(\text{aPl}(d, n)^\times)$  that also follows from Lemma 3.4, this  $\text{w}S_\lambda$  corresponds to the  $\text{wD}$ -equivariant cohomology class associated to  $\text{a}\Omega_\lambda$ . By Proposition 2.13, those classes form a  $\mathbb{Q}$ -basis of  $H^*(\text{wGr}(d, n))$ . The structure constants  $\text{w}c_{\lambda\mu}^\nu$  of  $H^*(\text{wGr}(d, n))$  are defined with respect to this basis  $\{\text{w}S_\lambda\}_\lambda$ . Since  $\zeta^*$  is the ring homomorphism given by

$$H_{\text{wR}}^*(\text{wGr}(d, n)) \rightarrow H_{\text{wR}}^*(\text{wGr}(d, n)) \otimes_{\mathbb{Q}} \mathbb{Q}[\text{wR}^*] \cong H^*(\text{wGr}(d, n)),$$

these non-equivariant structure constants are obtained by evaluating  $\text{w}\tilde{c}_{\lambda\mu}^\nu$  at  $wu_1 = \cdots = wu_{n-1} = 0$ , i.e.

$$\text{w}c_{\lambda\mu}^\nu = \text{w}\tilde{c}_{\lambda\mu}^\nu(wu_1 = \cdots = wu_{n-1} = 0).$$

In particular, the structure constants  $c_{\lambda\mu}^\nu$  of  $H^*(\text{Gr}(d, n))$  with respect to the ordinary Schubert classes  $S_\lambda$ , that are computed in [17, Theorem 1] also in terms of puzzles, can be obtained from  $\tilde{c}_{\lambda\mu}^\nu$  evaluating at  $u_1 = \cdots = u_{n-1} = 0$ . Here we recall that  $\tilde{c}_{\lambda\mu}^\nu$  is a polynomial in  $u_i$ 's where  $\{u_i = y_{i+1} - y_i, i = 1, \dots, n-1\}$  is a basis of  $\text{Lie}(R)_{\mathbb{Z}}^*$ .

**Corollary 5.6.** *Let  $\lambda, \mu, \nu \in \{d\}^n$ . The structure constant  $wc_{\lambda\mu}^\nu$  is given by*

$$wc_{\lambda\mu}^\nu = c_{\lambda\mu}^\nu + \sum_{\substack{\nu \rightarrow \nu^1 \rightarrow \dots \rightarrow \nu^k \geq \lambda, \mu}} \frac{\tilde{c}_{\lambda\mu}^{\nu^k}(u_i = w_{i+1} - w_i, i = 1, \dots, n-1)}{w_{\nu^1} \cdots w_{\nu^k}},$$

if  $l(\lambda) + l(\mu) = l(\nu)$  and is 0 otherwise. If  $w_1 \leq w_2 \leq \dots \leq w_n$ ,  $wc_{\lambda\mu}^\nu$  is non-negative for all  $\lambda, \mu, \nu \in \{d\}^n$ .

*Proof.* After the evaluation, the first summation of 5.8 vanishes unless  $l(\lambda) + l(\mu) - l(\nu) = 0$  since only the puzzles  $P$  without equivariant pieces can survive. Therefore, if  $l(\lambda) + l(\mu) = l(\nu)$ , by [17, Theorem 1] the first sum in the RHS of (5.8) becomes

$$\sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{\lambda\mu}^\nu \\ \text{no equivariant pieces}}} 1 = c_{\lambda\mu}^\nu.$$

In the second sum, only the puzzles  $Q$  with exactly  $k$  equivariant pieces survive after the evaluation, and so the summation vanishes unless  $l(\lambda) + l(\mu) - l(\nu) = 0$ . Therefore the second term becomes, if  $l(\lambda) + l(\mu) = l(\nu)$ ,

$$\begin{aligned} & \sum_{\substack{\nu \rightarrow \nu^1 \rightarrow \dots \rightarrow \nu^k \geq \lambda, \mu}} \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{\lambda\mu}^{\nu^k}}} \text{wt}^{\nu^k, \dots, \nu^1, \nu}(Q) \Big|_{wu_1 = \dots = wu_{n-1} = 0} \\ &= \sum_{\substack{\nu \rightarrow \nu^1 \rightarrow \dots \rightarrow \nu^k \geq \lambda, \mu}} \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{\lambda\mu}^{\nu^k}}} \frac{w(p_1)}{w_{\nu^1}} \cdots \frac{w(p_k)}{w_{\nu^k}} \\ &= \sum_{\substack{\nu \rightarrow \nu^1 \rightarrow \dots \rightarrow \nu^k \geq \lambda, \mu}} \frac{\tilde{c}_{\lambda\mu}^{\nu^k}(u_i = w_{i+1} - w_i, i = 1, \dots, n-1)}{w_{\nu^1} \cdots w_{\nu^k}}. \end{aligned}$$

Combining these terms, we obtain the desired formula. The positivity is a direct consequence of the equivariant positivity (Theorem 5.4).  $\square$

**Remark 5.7.** We can say that our positivity theorem holds for all weighted Grassmannians in a sense as follows: for a given  $w\text{Gr}(d, n)$  with the weight  $w = (w_1, \dots, w_n)$ , we can always perform a permutation on the basis  $\{e_1, \dots, e_n\}$  of  $\mathbb{C}^n$  so that the new order on the weight is non-decreasing. Then we can re-define the Schubert classes  $\{w\tilde{S}_\lambda\}_\lambda$  to make sure that the structure constants are positive.

We conclude this section by listing the *equivariant weighted Pieri rule* and working out a few examples. First, by Equation (5.5) interpreted through  $wh^*$  and  $w\kappa^*$ , we obtain the restriction of  $w\tilde{S}_{\text{div}}$  to the fixed points:

$$w\tilde{S}_{\text{div}}|_\lambda = y_{\text{id}} - \frac{w_{\text{id}}}{w_\lambda} y_\lambda.$$

Then we apply the translation formula in Proposition 5.2 to the usual equivariant Pieri rule (5.7) and obtain the *equivariant weighted Pieri rule*:

**Lemma 5.8.**

$$w\tilde{S}_{\text{div}}w\tilde{S}_\lambda = (w\tilde{S}_{\text{div}}|_\lambda)w\tilde{S}_\lambda + \sum_{\lambda' \rightarrow \lambda} \frac{w_{\text{id}}}{w_\lambda} w\tilde{S}_{\lambda'}.$$

**Remark 5.9.** From the equivariant weighted Pieri rule, it is easy to show a recursive formula for the structure constants  $w\tilde{c}_{\lambda\mu}^\nu$ , in the exactly same way shown in [16, Theorem 3]:

$$(5.10) \quad \left( w\tilde{S}_{\text{div}|\nu} - w\tilde{S}_{\text{div}|\lambda} \right) w\tilde{c}_{\lambda\mu}^\nu = \left( \sum_{\lambda' \rightarrow \lambda} \frac{w_{\text{id}}}{w_\lambda} w\tilde{c}_{\lambda'\mu}^\nu - \sum_{\nu \rightarrow \nu'} \frac{w_{\text{id}}}{w_{\nu'}} w\tilde{c}_{\lambda\mu}^{\nu'} \right).$$

However this equation (5.10) plays no role in the derivation of our main formula, while the recursive formula in [16] plays a crucial role in their process of obtaining the original puzzle formula for  $\tilde{c}_{\lambda\mu}^\nu$ .

**Example 5.10** (Weighted Projective Space  $w\text{Gr}(1, n)$ ). By definition,  $a\text{Pl}(1, n)^\times = \mathbb{C}^n \setminus \{0\}$ . The weighted diagonal is

$$wD_{\mathbb{C}} = \{(t^{w_1+a}, \dots, t^{w_n+a}) \in (\mathbb{C}^\times)^n \mid t \in \mathbb{C}^\times\}.$$

Therefore  $w\text{Gr}(1, n)$  is the weighted projective space  $\mathbb{CP}_{b_1, \dots, b_n}$  of weights  $b_i = w_i + a$ . Moreover the Schubert cell decomposition of  $a\text{Pl}(1, n)^\times$  is

$$\mathbb{C}^n \setminus \{0\} = a\Omega_{\{n\}}^\circ \coprod \cdots \coprod a\Omega_{\{1\}}^\circ$$

where  $a\Omega_{\{k\}} = \mathbb{C}^{k-1} \times \mathbb{C}^\times \times \{0\}^{n-k}$ . Thus the weighted Schubert varieties are smaller weighted projective spaces embedded naturally in  $\mathbb{CP}_{b_1, \dots, b_n}$ :

$$\mathbb{CP}_{b_1, \dots, b_n} \supset \mathbb{CP}_{b_1, \dots, b_{n-1}} \supset \cdots \supset \mathbb{CP}_{b_1, b_2, b_3} \supset \mathbb{CP}_{b_1, b_2} \supset \text{pt}.$$

The GKM description in Proposition 4.3 agrees with the one given in [20, Section 6] and it coincides with the Stanley-Reisner ring of the boundary of the  $(n-1)$ -simplex:

$$H_T^*(a\text{Pl}(1, n)^\times) = \mathbb{Q}[y_1, \dots, y_n] / (y_1 \cdots y_n).$$

With this identification, the Schubert classes  $a\tilde{S}_\lambda$  are given by

$$(5.11) \quad a\tilde{S}_{\{n\}} = 1, \quad a\tilde{S}_{\{n-1\}} = y_n, \quad \dots, \quad a\tilde{S}_{\{k\}} = y_{k+1} \cdots y_n, \quad \dots, \quad a\tilde{S}_{\{1\}} = y_2 \cdots y_n.$$

where the Bruhat order is  $\{n\} \leq \cdots \leq \{1\}$ . The equivariant weighted Pieri rule gives

$$w\tilde{S}_{\{n-1\}} \cdot w\tilde{S}_{\{k\}} = \left( y_n - \frac{b_n}{b_k} y_k \right) w\tilde{S}_{\{k\}} + \frac{b_n}{b_k} w\tilde{S}_{\{k-1\}}.$$

This is actually obvious in the presentation  $\mathbb{Q}[y_1, \dots, y_n] / (y_1 \cdots y_n)$ :

$$y_n \cdot (y_{k+1} \cdots y_n) = \left( y_n - \frac{b_n}{b_k} y_k \right) (y_{k+1} \cdots y_n) + \frac{b_n}{b_k} y_k \cdots y_n.$$

**Example 5.11** (Relation to the work of Kawasaki [13]). In this example, all cohomologies are over  $\mathbb{Z}$ -coefficients. The integral cohomology of the weighted projective space  $H^*(\mathbb{CP}_b; \mathbb{Z})$  is known to be a free  $\mathbb{Z}$ -module by the work of Kawasaki [13]. Namely, the map

$$\zeta : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{C}^n \setminus \{0\}, \quad (z_1, \dots, z_n) \mapsto (z_1^{b_1}, \dots, z_n^{b_n})$$

induces a map  $\bar{\zeta} : \mathbb{CP}^{n-1} \rightarrow \mathbb{CP}_b$  and the inclusion  $\bar{\zeta}^* : H^*(\mathbb{CP}_b; \mathbb{Z}) \rightarrow H^*(\mathbb{CP}; \mathbb{Z})$ . Let  $u_1, \dots, u_{n-1}$  be a  $\mathbb{Z}$ -basis of  $\text{Lie}(R)_{\mathbb{Z}}^*$ . If we represent  $H^*(\mathbb{CP}; \mathbb{Z})$  as

$$\frac{\mathbb{Z}[y_1, \dots, y_n]}{(y_1 \cdots y_n, u_1, \dots, u_{n-1})}$$

following [5], and identify  $H^*(\mathbb{CP}_b; \mathbb{Z})$  with the image of  $\zeta^*$ , then  $H^*(\mathbb{CP}_b; \mathbb{Z})$  is, as a free  $\mathbb{Z}$ -module, generated by

$$\gamma_1 := l_1^b = 1, \quad \gamma_2 := l_2^b y_n, \quad \gamma_3 := l_3^b y_{n-1} y_n, \quad \dots, \quad \gamma_n := l_n^b y_2 \cdots y_n$$

where

$$l_k^b := \text{l.c.m. of } \left\{ \frac{b_{i_1} \cdots b_{i_k}}{\gcd(b_{i_1}, \dots, b_{i_k})} \mid 1 \leq i_1 < \dots < i_k \leq n \right\}.$$

On the other hand,  $\zeta$  and the homomorphism  $D \rightarrow \text{w}D$  defined by  $(t, \dots, t) \mapsto (t^{b_1}, \dots, t^{b_n})$  induce a map  $\zeta' : ED \times_D (\mathbb{C}^n \setminus \{0\}) \rightarrow E\text{w}D \times_{\text{w}D} (\mathbb{C}^n \setminus \{0\})$ . The cohomology  $H_{\text{w}D}^*(\mathbb{C}^n \setminus \{0\})$  is known to be

$$\frac{\mathbb{Z}[y_1, \dots, y_n]}{(y_1 \cdots y_n, u_1^b, \dots, u_{n-1}^b)}$$

where  $\{u_i^b\}$  is a  $\mathbb{Z}$ -basis of  $\text{Lie}(\text{w}R)_{\mathbb{Z}}^* \subset \text{Span}_{\mathbb{Z}}\{y_1, \dots, y_n\}$  (c.f. Example 6.1 [19]) and the pullback  $\zeta'^*$  sends  $y_i$  to  $b_i y_i$ . Our *non-equivariant* Schubert classes  $\text{a}S_{\{k\}}$  live in  $H_{\text{w}D}^*(\mathbb{C}^n \setminus \{0\}; \mathbb{Z})$  as the monomial  $y_{k+1} \cdots y_n$ . Since  $\tilde{\zeta}^*$  factors through  $\zeta'^*$  and  $H_{\text{w}D}^*(\mathbb{C}^n \setminus \{0\}; \mathbb{Z})$  has no  $\mathbb{Z}$ -torsions in the degrees between 0 and  $2(n-1)$  (see Theorem 4.2 [11]), the pullbacks of the Kawasaki's basis along the projection  $\pi : E\text{w}D \times_{\text{w}D} \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{CP}^b$  are the following multiples of our Schubert classes:

$$\pi^*(\gamma_1) = \text{a}S_{\{n\}} \quad \text{and} \quad \pi^*(\gamma_k) = \frac{l_k^b}{b_{n-k+2} b_{n-k+3} \cdots b_n} \text{a}S_{\{n-k+1\}}, \quad k = 2, \dots, n.$$

**Example 5.12** ( $\text{wGr}(2, 4)$ ). Here we demonstrate the computation of the product  $\text{w}\tilde{S}_{23} \text{w}\tilde{S}_{23}$ . By the upper triangularity of the GKM description of  $\text{w}\tilde{S}_{23}$ , the product must be written by

$$\text{w}\tilde{S}_{23} \text{w}\tilde{S}_{23} = \text{w}\tilde{c}_{23,23}^{23} \text{w}\tilde{S}_{23} + \text{w}\tilde{c}_{23,23}^{13} \text{w}\tilde{S}_{13} + \text{w}\tilde{c}_{23,23}^{12} \text{w}\tilde{S}_{12},$$

where

$$\begin{aligned} \text{w}\tilde{c}_{23,23}^{23} &= \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{23,23}^{23}}} \text{wt}^{23}(P) \\ \text{w}\tilde{c}_{23,23}^{13} &= \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{23,23}^{13}}} \text{wt}^{13}(P) + \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{23,23}^{23}}} \text{wt}^{23,13}(Q) \\ \text{w}\tilde{c}_{23,23}^{12} &= \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{23,23}^{12}}} \text{wt}^{12}(P) + \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{23,23}^{13}}} \text{wt}^{13,12}(Q) + \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{23,23}^{23}}} \text{wt}^{23,13,12}(Q). \end{aligned}$$

We can compute the above from the product for ordinary Grassmannian

$$\tilde{S}_{23} \tilde{S}_{23} = (y_4 - y_2)(y_4 - y_3) \tilde{S}_{23} + (y_4 - y_3) \tilde{S}_{13} + \tilde{S}_{12};$$

or equivalently by the fact that:

- there is exactly one puzzle  $P$  such that  $\partial P = \Delta_{23,23}^{23}$  with two equivariant pieces  $p_1$  and  $p_2$  with the weights  $\text{wt}(p_1) = y_4 - y_3$  and  $\text{wt}(p_2) = y_4 - y_2$ ;
- there is exactly one puzzle  $P$  such that  $\partial P = \Delta_{23,23}^{13}$  with a equivariant piece with the weight  $y_4 - y_3$ ;
- there is exactly one puzzle  $P$  such that  $\partial P = \Delta_{23,23}^{12}$  without equivariant pieces.

Here are the computation:

$$\begin{aligned}
w\tilde{c}_{23,23}^{23} &= \left( y_4 - y_2 - (w_4 - w_2) \frac{y_{23}}{w_{23}} \right) \left( (y_4 - y_3) - (w_4 - w_3) \frac{y_{23}}{w_{23}} \right) \\
w\tilde{c}_{23,23}^{13} &= (y_4 - y_3) - (w_4 - w_3) \frac{y_{13}}{w_{13}} + \frac{w_4 - w_2}{w_{23}} \left( (y_4 - y_3) - (w_4 - w_3) \frac{y_{13}}{w_{13}} \right) \\
&\quad + \left( (y_4 - y_2) - (w_4 - w_2) \frac{y_{23}}{w_{23}} \right) \frac{w_4 - w_3}{w_{23}} \\
w\tilde{c}_{23,23}^{12} &= 1 + \frac{w_4 - w_3}{w_{13}} + \frac{w_4 - w_2}{w_{23}} \frac{w_4 - w_3}{w_{13}}
\end{aligned}$$

Similarly we can also work out

$$w\tilde{S}_{23}w\tilde{S}_{14} = w\tilde{c}_{23,14}^{13}w\tilde{S}_{13} + w\tilde{c}_{23,14}^{12}w\tilde{S}_{12}$$

from  $\tilde{S}_{23}\tilde{S}_{14} = (y_4 - y_1)\tilde{S}_{13}$ :

$$\begin{aligned}
w\tilde{c}_{23,14}^{13} &= \sum_{\substack{\text{puzzle } P \\ \partial P = \Delta_{23,14}^{13}}} \text{wt}^{13}(P) = (y_4 - y_1) - (w_4 - w_1) \frac{y_{13}}{w_{13}} \\
w\tilde{c}_{23,14}^{12} &= \sum_{\substack{\text{puzzle } Q \\ \partial Q = \Delta_{23,14}^{13}}} \text{wt}^{13,12}(Q) = \frac{w_4 - w_1}{w_{13}}.
\end{aligned}$$

## 6. Factorial Schur Functions

We discuss the relation between equivariant weighted Schubert classes and the factorial Schur functions. More precisely, we show that the restriction  $w\tilde{S}_\lambda|_\mu$  of the weighted Schubert classes can be obtained by specializing the factorial Schur functions.

Let  $x = (x_1, \dots, x_d)$  and  $a = (a_i)_{i \in \mathbb{Z}}$  be sequences of variables. Let

$$(x_j|a)^k := (x_j - a_1) \cdots (x_j - a_k) \quad (1 \leq j \leq d).$$

The Young diagram  $\underline{\lambda}$  corresponds to each  $\lambda \in \{n_d\}$  by setting the number of boxes in the  $i$ -th row to be  $\underline{\lambda}^i := n - d + i - \lambda_i$  where  $i = 1, \dots, d$ . The *factorial Schur function* corresponding to  $\lambda$  (c.f. [21]) is defined by

$$(6.1) \quad s_\lambda(x|a) := \frac{\det \left[ (x_j|a)^{\underline{\lambda}^i + d - i} \right]_{1 \leq i, j \leq d}}{\prod_{i < j} (x_i - x_j)}.$$

For any sequence  $b = (b_i)_{i \in \mathbb{Z}}$ , let  $\bar{b} = (\bar{b}_i)_{i \in \mathbb{Z}}$  be defined by  $\bar{b}_i := b_{n+1-i}$ . For each  $\mu \in \{n_d\}$ , let

$$(6.2) \quad b(\mu) = (b_{\mu_1}, \dots, b_{\mu_d}).$$

The vanishing theorem ([22], [21, Theorem 2.1], [16, Section 6]) shows that the restriction of the equivariant Schubert class  $\tilde{S}_\lambda$  to  $[e_\mu]$  is given by

$$(6.3) \quad \tilde{S}_\lambda|_\mu = s_\lambda(-y(\mu) | -\bar{y}).$$

To generalize this equation to the weighted Schubert classes, we introduce the  $\mu$ -shifted sequence associated to each sequence  $b = (b_i)_{i \in \mathbb{Z}}$  by

$$b^\mu := \left( b_i - w_i \frac{b_\mu}{w_\mu} \right)_{i \in \mathbb{Z}} \quad \text{where } b_\mu = \sum_{k \in \mu} b_k.$$



**Theorem 6.1.** For all  $\lambda, \mu \in \{d\}^n$ , we have

$$\mathrm{w}\tilde{S}_\lambda|_\mu = s_\lambda(-y^\mu(\mu)| - \overline{y^\mu}).$$

*Proof.* We rewrite (6.3) as

$$\tilde{S}_\lambda|_\mu \times \prod_{i < j} (-y_{\mu_i} + y_{\mu_j}) = \det \left[ (-y_{\mu_j} | - \overline{y})^{\Delta^i + d - i} \right]_{1 \leq i, j \leq d}.$$

Now recall the diagram (4.5). By the isomorphism  $\kappa_\mu^*$ , we can regard this equality as in  $\mathbb{Q}[T^*]/(y_\mu)$  so that we can shift it by multiples of  $y_\mu$  to obtain

$$\mathrm{a}\tilde{S}_\lambda|_\mu \times \prod_{i < j} (-(y^\mu)_{\mu_i} + (y^\mu)_{\mu_j}) = \det \left[ \prod_{p=1}^{\Delta^i + d - i} (-(y^\mu)_{\mu_j} + (y^\mu)_{n+1-p}) \right]_{1 \leq i, j \leq d}.$$

Since  $-(y^\mu)_{\mu_i} + (y^\mu)_{\mu_j}$  and  $-(y^\mu)_{\mu_j} + (y^\mu)_{n+1-p}$  are elements of  $\mathbb{Q}[\mathrm{w}R^*]$ , this becomes, under the isomorphism  $\mathrm{w}\kappa_\mu^*$ ,

$$\mathrm{w}\tilde{S}_\lambda|_\mu \times \prod_{i < j} (-(y^\mu)_{\mu_i} + (y^\mu)_{\mu_j}) = \det [(-(y^\mu)_{\mu_j} | - \overline{y^\mu})^{\Delta^i + d - i}]_{1 \leq i, j \leq d},$$

which is the desired equation.  $\square$

**Remark 6.2.** In fact,  $\mathrm{w}\tilde{S}_\lambda|_\mu$  can also be obtained by specializing the *weighted factorial Schur functions* that will be introduced and studied in a subsequent paper.

## 7. Appendix

**7.1. Proof of Proposition 2.7.** First, we show  $\mathrm{a}\Omega_\lambda \subset \prod_{\mu \geq \lambda} \mathrm{a}\Omega_\mu^\circ$ . Let  $x \in \mathrm{a}\Omega_\lambda$ . Then there exists a sequence  $\{x_N\}_{N=0}^\infty \subset \mathrm{a}\Omega_\lambda^\circ$  such that  $x_N$  converges to  $x$  as  $N$  goes to  $\infty$ . By Lemma 2.3,  $(x_N)_\eta = 0$  for all  $\eta \not\geq \lambda$ . Therefore  $x_\eta = 0$  for all  $\eta \not\geq \lambda$ , i.e.  $x \notin \mathrm{a}\Omega_\eta^\circ$  for all  $\mu \not\geq \lambda$ . By the decomposition (2.6) of  $\mathrm{aPl}(d, n)^\times$ , we obtain  $x \in \prod_{\lambda \leq \mu} \mathrm{a}\Omega_\mu^\circ$ . Next, we show  $\mathrm{a}\Omega_\lambda \supset \prod_{\mu \geq \lambda} \mathrm{a}\Omega_\mu^\circ$ . If  $\mu \not\geq \lambda$ , then there is a covering sequence  $\mu = \mu^s \rightarrow \mu^{s-1} \rightarrow \cdots \rightarrow \mu^1 \rightarrow \lambda$  where  $s = l(\mu) - l(\lambda)$ . Thus it suffices to show that  $\mathrm{a}\Omega_\lambda \supset \mathrm{a}\Omega_\mu^\circ$  for any  $\mu$  such that  $\mu \rightarrow \lambda$ , i.e. for some  $1 \leq p \leq d$ ,

$$\mu_p = \lambda_p - 1 \quad \text{and} \quad \mu_q = \lambda_q \quad \text{for all} \quad q \neq p.$$

Let  $y \in \mathrm{a}\Omega_\mu^\circ$ . We construct a sequence  $\{x_N\}_{N \in \mathbb{N}} \subset \mathrm{a}\Omega_\lambda^\circ$  which converges to  $y$  as  $N$  goes to  $\infty$ . For brevity, we omit the index  $N$  and write  $x_N = x$ . Since any point in  $\mathrm{a}U^\lambda$  is determined by its coordinates of the indexes in  $\{\lambda\} \coprod [\lambda]$ , we define  $x$  to be the element of  $\mathrm{a}U^\lambda$  uniquely given by

$$\begin{aligned} x_\lambda &:= N^{-1}, \\ x_\nu &:= y_\nu \quad \text{for all } \nu \in [\lambda] \cap (\{\mu\} \coprod [\mu]), \\ x_{\rho_{q,\alpha}} &:= \left\{ y(\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \alpha) x(\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \lambda_p) \right. \\ &\quad \left. - y(\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \alpha) y(\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \lambda_p) \right\} \frac{(-1)^{\delta+r}}{y_\mu} \end{aligned}$$

where  $\delta = 0$  if  $p < q$  and  $\delta = 1$  if  $q < p$ , and  $x_{\rho_{q,\alpha}} = (-1)^r x(\mu_1, \dots, \check{\mu}_q, \dots, \check{\mu}_p, \dots, \mu_d, \alpha, \lambda_p)$  for some integer  $r$  in the extended notation given at Section 2.2. Here we have used the decomposition of  $[\lambda]$  into  $[\lambda] \cap (\{\mu\} \coprod [\mu])$  and

$$[\lambda] \setminus (\{\mu\} \coprod [\mu]) = \{ \rho_{q,\alpha} := \{\lambda_1, \dots, \lambda_d, \alpha\} \setminus \{\lambda_q\} \mid q \neq p \text{ and } \alpha \notin \lambda \cup \{\mu_p\} \}.$$

This  $x$  is an element of  $\mathrm{a}\Omega_\lambda^\circ$  since  $x_\eta = 0$  for all  $\eta \in [\lambda]_-$  by Lemma 2.3. Indeed, if  $\eta \in [\lambda] \cap (\{\mu\} \coprod [\mu])$ , then  $\eta \in [\mu]_-$  so that  $x_\eta = y_\eta = 0$  by  $y \in \mathrm{a}\Omega_\mu^\circ$ . If  $\eta = \rho_{q,\alpha}$ , then  $\mu_q = \lambda_q < \alpha$ , i.e.  $\{\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \alpha\} \in [\mu]_-$ . Hence, the first term of  $x_{\rho_{q,\alpha}}$  vanishes. Moreover,

if  $\mu_p < \alpha$ , we have  $\{\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \alpha\} \in [\mu]_-$ , and if not,  $\mu_q < \alpha < \mu_p < \lambda_p$ , we have  $\{\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \lambda_p\} \in [\mu]_-$ . In both cases, the second term of  $x_{\rho_{q,\alpha}}$  vanishes.

Since  $x \in aU^\mu$  by  $x_\mu = y_\mu \neq 0$ , we can compare  $x$  and  $y$  under the chart  $\psi_\mu$ . In fact,  $\psi_\mu(x)$  and  $\psi_\mu(y)$  coincide except for the  $\lambda$ -component, therefore  $x$  goes to  $y$  when  $N$  goes to  $\infty$ , as desired. By the definition of  $x$ , it suffices to check that  $x_\xi = y_\xi$  for any  $\xi \in (\{\mu\} \coprod [\mu]) \setminus (\{\lambda\} \coprod [\lambda])$ . Observe that  $\xi$  can be written as  $\{\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \alpha\}$  where  $q \neq p$  and  $\alpha \notin \lambda \cup \{\mu_p\}$ . By the Plücker relation for the sequences  $\mu_1, \dots, \check{\mu}_q, \dots, \check{\mu}_p, \dots, \mu_d, \alpha$  and  $\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \lambda_p, \mu_p$ , we have

$$\begin{aligned} & x(\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \alpha) x(\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \lambda_p) \\ &= (-1)^{d-p+\delta} x(\mu_1, \dots, \check{\mu}_q, \dots, \check{\mu}_p, \dots, \mu_d, \alpha, \mu_p) x(\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \lambda_p) \\ &= x(\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \alpha) x(\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \lambda_p) \\ &\quad + (-1)^\delta x(\mu_1, \dots, \check{\mu}_q, \dots, \check{\mu}_p, \dots, \mu_d, \alpha, \lambda_p) x(\mu_1, \dots, \mu_d) \\ &= y(\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \alpha) y(\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \lambda_p) + (-1)^{\delta+r} x_{\rho_{q,\alpha}} y_\mu \\ &= y(\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \alpha) x(\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \lambda_p) \end{aligned}$$

where we used the definition of  $x_{\rho_{q,\alpha}}$  at the last equality. So we obtain

$$x(\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \alpha) = y(\mu_1, \dots, \check{\mu}_q, \dots, \mu_d, \alpha)$$

since  $x(\mu_1, \dots, \check{\mu}_p, \dots, \mu_d, \lambda_p) = \pm x_\lambda \neq 0$ .  $\square$

**7.2. Proof of (2.10).** Let us denote  $\overline{H}_*$  the Borel-Moore homology over  $\mathbb{Q}$ . We first quote a lemma which will be used in the proof of the next proposition.

**Lemma 7.1.** *Let  $D^m$  be the closed unit disc in  $\mathbb{R}^m$ . Let  $\iota : \{0\} \hookrightarrow D^m$  be the inclusion. Then  $\iota_* : \overline{H}_*(\{0\}) \rightarrow \overline{H}_*(D^m)$  is an isomorphism.*

Let  $a_1, \dots, a_m, b$  be positive integers. Then  $\mathbb{C}^\times$  acts on  $\mathbb{C}^m$  by

$$g \cdot (z_1, \dots, z_m) = (g^{a_1} z_1, \dots, g^{a_m} z_m)$$

for any  $g \in \mathbb{C}^\times$  and  $(z_1, \dots, z_m) \in \mathbb{C}^m$ . Let  $G := \{g \in \mathbb{C}^\times \mid g^b = e\}$ . Observe that the Borel-Moore homology  $\overline{H}_i(\mathbb{C}^m/G)$  is defined ( $\mathbb{C}^m/G$  can be realized by an open subset of  $w\mathbb{P}(a_1, \dots, a_m, b)$  which can also be realized as a closed subset of a projective space  $\mathbb{P}^N$  for some integer  $N$ ).

**Proposition 7.2.**

$$\overline{H}_i(\mathbb{C}^m/G) \cong \begin{cases} \mathbb{Q} & (i = 2m) \\ 0 & (\text{otherwise}) \end{cases}$$

*Proof.* Let  $D^{2m} = \{z \in \mathbb{C}^m \mid |z| \leq 1\}$ . Then the  $G$ -action on  $\mathbb{C}^m$  restricts on  $\text{int} D^{2m}$  since  $G \subset S^1 \subset \mathbb{C}^\times$ , and the  $G$ -equivariant homeomorphism  $\text{int} D^{2m} \cong \mathbb{C}^m$  defined by

$$z \mapsto \left( z_1 / \sqrt{1 - |z|^2}, \dots, z_m / \sqrt{1 - |z|^2} \right)$$

induces an homeomorphism  $\text{int} D^{2m}/G \cong \mathbb{C}^m/G$ . We calculate  $\overline{H}_i(\text{int} D^{2m}/G)$  in the following.

The Borel-Moore homology  $\overline{H}_*(D^{2m}/G)$  is also defined because  $D^{2m}/G$  is a closed subset of  $\mathbb{C}^m/G$ . Then, we have an exact sequence associated to the open embedding  $\text{int} D^{2m}/G \hookrightarrow D^{2m}/G$

$$\cdots \rightarrow \overline{H}_i(D^{2m}/G) \rightarrow \overline{H}_i(\text{int} D^{2m}/G) \rightarrow \overline{H}_{i-1}(S^{2m-1}/G) \xrightarrow{\iota_*} \overline{H}_{i-1}(D^{2m}/G) \rightarrow \cdots$$

where  $\iota : S^{2m-1} \hookrightarrow D^{2m}$  is the inclusion. Since the spaces  $D^{2m}/G$  and  $S^{2m-1}/G$  are compact, locally contractible spaces, we have

$$\begin{aligned}\overline{H}_i(D^{2m}/G) &\cong H_i(D^{2m}/G) \cong \begin{cases} \mathbb{Q} & (i = 0) \\ 0 & (\text{otherwise}), \end{cases} \\ \overline{H}_i(S^{2m-1}/G) &\cong H_i(S^{2m-1}/G) \cong \begin{cases} \mathbb{Q} & (i = 0, 2m-1) \\ 0 & (\text{otherwise}). \end{cases}\end{aligned}$$

(see [23, Lem.14, sec.10, chap.6]). Hence, the above exact sequence shows

$$\overline{H}_i(\text{int} D^{2m}/G) \cong \begin{cases} \mathbb{Q} & (i = 2m) \\ 0 & (i \neq 0, 1, 2m). \end{cases}$$

We prove  $\overline{H}_0(\text{int} D^{2m}/G) = \overline{H}_1(\text{int} D^{2m}/G) = 0$ . Since we have

$$0 \rightarrow \overline{H}_1(\text{int} D^{2m}/G) \rightarrow \overline{H}_0(S^{2m-1}/G) \xrightarrow{\iota_*} \overline{H}_0(D^{2m}/G) \rightarrow \overline{H}_0(\text{int} D^{2m}/G) \rightarrow 0,$$

it suffices to show that  $\overline{H}_0(S^{2m-1}/G) \xrightarrow{\iota_*} \overline{H}_0(D^{2m}/G)$  is an isomorphism. Recall that  $D^{2m}/G$  is compact and can be embedded into  $\mathbb{R}^M$  for some  $M$ . Without loss of generality, we can assume that there is a sequence of closed embeddings

$$\{0\} \rightarrow S^{2m-1}/G \hookrightarrow D^{2m}/G \rightarrow D(\hookrightarrow \mathbb{R}^M)$$

such that the compositions of embeddings coincides with the natural inclusion  $j : \{0\} \hookrightarrow D$ , where  $D$  is the closed unit disk in  $\mathbb{R}^M$  and 0 is the origin of  $\mathbb{R}^M$ . Out of this sequence of closed embeddings, we obtain

$$\overline{H}_0(\{0\}) \rightarrow \overline{H}_0(S^{2m-1}/G) \rightarrow \overline{H}_0(D^{2m}/G) \rightarrow \overline{H}_0(D)$$

which coincides with  $j_* : \overline{H}_0(\{0\}) \rightarrow \overline{H}_0(D)$ . Since all the entries in this sequence are isomorphic to  $\mathbb{Q}$ , it is enough to show that  $j_*$  is an isomorphism which is proved by Lemma 7.1.  $\square$

**7.3. The GKM description of  $H_{\text{wR}}^*(\text{wGr}(d, n))$ .** We give an alternative proof of Proposition 4.2 as a direct consequence of [9, Theorem 7.2], by studying 0 and 1 dimensional orbits and Lie algebras of isotropy subgroups of  $\text{wR}$ .

We start with notations. For  $\lambda \in \{d\}^n$  and  $\gamma \in \text{Lie}(T)$ , we denote  $\gamma_\lambda := \sum_{i \in \lambda} \gamma_i$ . Since  $\text{Lie}(\text{wR}) = \text{Lie}(T)/\text{Lie}(\text{wD})$ , we write an element of  $\text{Lie}(\text{wR})$  as  $[\gamma]$  where  $\gamma \in \text{Lie}(\text{wR})$ . Define

$$\mathcal{O}_{\lambda\mu} := \{[x] \in \text{wP}(\wedge^d \mathbb{C}^n) \mid x(\lambda) \neq 0, x(\mu) \neq 0, x(\eta) = 0 (\eta \neq \lambda, \mu)\}.$$

Let  $\mathbb{P}^1(w_\lambda, w_\mu)$  be the weighted projective line with weight  $w_\lambda$  and  $w_\mu$ . Consider a continuous map  $f : \mathbb{C}^2 \setminus \{0\} \rightarrow \text{aPl}(d, n)^\times$  defined by

$$f(x, y)(\eta) = \begin{cases} x & (\text{if } \eta = \lambda) \\ y & (\text{if } \eta = \mu) \\ 0 & (\text{otherwise}). \end{cases}$$

Then the map  $f$  induces a continuous map  $\overline{f} : \mathbb{P}^1(w_\lambda, w_\mu) \rightarrow \text{wGr}(d, n)$  which is a homeomorphism onto  $\overline{\mathcal{O}_{\lambda\mu}} = \mathcal{O}_{\lambda\mu} \cup \{[e_\lambda], [e_\mu]\}$ .

For brevity, let  $X := \text{wGr}(d, n)$ , and denote

$$\begin{aligned}X_0 &:= \{[x] \in \text{wGr}(d, n) \mid \text{corank } \text{wR}_{[x]} = 0\}, \\ X_1 &:= \{[x] \in \text{wGr}(d, n) \mid \text{corank } \text{wR}_{[x]} \leq 1\}.\end{aligned}$$

where  $\mathrm{wR}_{[x]}$  is the isotropy subgroup of  $\mathrm{wR}$  at  $[x]$  and  $\mathrm{corank} \mathrm{wR}_{[x]} := (n-1) - \mathrm{rank} \mathrm{wR}_{[x]}$ . In other words,  $X_0$  is the set of  $\mathrm{wR}$ -fixed points, and  $X_1$  is the set of 0 and 1 dimensional orbits of  $\mathrm{wR}$ . The data in the next proposition will provide us the GKM description of  $H_{\mathrm{wR}}^*(\mathrm{wGr}(d, n))$  (Proposition 4.2).

**Proposition 7.3.** *For the  $\mathrm{wR}$ -action on  $\mathrm{wGr}(d, n)$ , the followings hold:*

- (1)  $X_0$  consists of the points  $e_\lambda$  for all  $\lambda \in \{ \frac{n}{d} \}$ .
- (2)  $X_1$  is the union of  $\overline{\mathcal{O}_{\lambda\mu}}$  for all  $\lambda$  and  $\mu$  such that  $|\lambda \cap \mu| = d-1$ .
- (3) For any  $[x] \in \mathcal{O}_{\lambda\mu}$  where  $|\lambda \cap \mu| = d-1$ ,

$$\mathrm{Lie}(\mathrm{wR}_{[x]}) = \{ [\gamma] \in \mathrm{Lie}(\mathrm{wR}) \mid w_\mu \gamma_\lambda - w_\lambda \gamma_\mu = 0 \}.$$

*Proof.* Let  $[x] \in \mathrm{wGr}(d, n)$ , and write  $\{ \lambda \in \{ \frac{n}{d} \} \mid x_\lambda \neq 0 \} = \{ \lambda^{(1)}, \dots, \lambda^{(p)} \}$ . We show

$$(7.1) \quad \mathrm{Lie}(\mathrm{wR}_{[x]}) = \{ [\gamma] \in \mathrm{Lie}(\mathrm{wR}) \mid (w_{\check{\lambda}^{(1)}} \cdots w_{\lambda^{(p)}}) \gamma_{\lambda^{(1)}} = \cdots = (w_{\lambda^{(1)}} \cdots w_{\check{\lambda}^{(p)}}) \gamma_{\lambda^{(p)}} \}.$$

By the definition of the isotropy subgroup, we have

$$(7.2) \quad \mathrm{wR}_{[x]} = \{ [t] \in \mathrm{wR} \mid \text{for some } \epsilon \in \mathbb{C}^\times, t_{\lambda^{(i)}} = \epsilon^{w_{\lambda^{(i)}}} (1 \leq i \leq p) \}.$$

Denoting

$$H := \{ [t] \in \mathrm{wR} \mid (t_{\lambda^{(1)}})^{w_{\check{\lambda}^{(1)}} w_{\lambda^{(2)}} \cdots w_{\lambda^{(p)}}} = \cdots = (t_{\lambda^{(p)}})^{w_{\lambda^{(1)}} \cdots w_{\lambda^{p-1}}} w_{\check{\lambda}^{(p)}} \},$$

we have  $\mathrm{wR}_{[x]} \subset H$  which implies  $\mathrm{Lie}(\mathrm{wR}_{[x]}) \subset \mathrm{Lie}(H)$ . Observe that  $\mathrm{Lie}(H)$  is equal to the right hand side of (7.1). For any  $\gamma \in \mathrm{Lie}(H)$ , putting  $E := t_{\lambda^{(1)}}/w_{\lambda^{(1)}}$ , we have  $t_{\lambda^{(i)}} = w_{\lambda^{(i)}} E$  ( $1 \leq i \leq p$ ). So, for any  $s \in \mathbb{C}$ , we have

$$(\exp(s\gamma))_{\lambda^{(i)}} = \exp(s\gamma_{\lambda^{(i)}}) = \exp(sw_{\lambda^{(i)}} E) = (\exp(sE))^{w_{\lambda^{(i)}}}$$

for all  $1 \leq i \leq p$ . This shows  $\exp(s\gamma) \in \mathrm{wR}_{[x]}$  for any  $s \in \mathbb{C}$  which implies  $\gamma \in \mathrm{Lie}(\mathrm{wR}_{[x]})$ . Hence, we obtain  $\mathrm{Lie}(\mathrm{wR}_{[x]}) = \mathrm{Lie} H$ , i.e. the equality (7.1). Now we see

$$\mathrm{corank} \mathrm{wR}_{[x]} = 0 \text{ if and only if } |\{ \lambda \in \{ \frac{n}{d} \} \mid x_\lambda \neq 0 \}| = 1,$$

$$\mathrm{corank} \mathrm{wR}_{[x]} = 1 \text{ if and only if } |\{ \lambda \in \{ \frac{n}{d} \} \mid x_\lambda \neq 0 \}| = 2.$$

Hence, we obtain the claims (1) and (3). For (2), let  $\lambda, \mu \in \{ \frac{n}{d} \}$ , and consider  $\mathcal{O}_{\lambda\mu}$ . Suppose that we have  $|\lambda \cap \mu| = d-1$ . Let us write  $\lambda = \{ \lambda_1, \dots, \lambda_d \} \subset [n]$  and  $\mu = \{ \lambda_1, \dots, \check{\lambda}_s, \dots, \lambda_d, \alpha \} \subset [n]$  for some  $\alpha \notin \lambda$ . Then any  $[x] \in \mathcal{O}_{\lambda\mu}$  can be written as

$$\begin{aligned} [x] &= [ae_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d} + be_{\lambda_1} \wedge \cdots \wedge e_{\check{\lambda}_s} \wedge \cdots \wedge e_{\lambda_d} \wedge e_\alpha] \\ &= [e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_{s-1}} \wedge (ae_{\lambda_s} + (-1)^{d-i} be_\alpha) \wedge e_{\lambda_{s+1}} \wedge \cdots \wedge e_{\lambda_d}] \end{aligned}$$

for some  $a, b \in \mathbb{C}^\times$ . Hence, we obtain  $\mathcal{O}_{\lambda\mu} \subset \mathrm{wGr}(d, n)$  since  $\mathrm{aPl}(d, n)^\times = \mathrm{Im} \wedge^d - \{0\}$ . On the other hand, suppose  $|\lambda \cap \mu| < d-1$ . For any element  $[x] \in \mathcal{O}_{\lambda\eta} \cap \mathrm{wGr}(d, n)$ , we have  $x_\xi = 0$  for any  $\xi \neq \lambda, \eta$  because of the definition of  $\mathcal{O}_{\lambda\eta}$ . The condition  $|\lambda \cap \eta| < d-1$  means that all the coordinates of the orbifold chart  $\overline{\psi}_\lambda$  of  $[x]$  are zero. This means that  $[x] = [e_\lambda]$  which contradicts to  $[x] \in \mathcal{O}_{\lambda\eta}$ , and we obtain  $\mathcal{O}_{\lambda\eta} \cap \mathrm{wGr}(d, n) = \emptyset$ .  $\square$

There exists a natural isomorphism between the Čech cohomology theory and the singular cohomology theory for any closed pair of locally contractible, paracompact, Hausdorff spaces. Thus, the results in [3] applies for the singular  $\mathrm{wR}$ -equivariant cohomology: the restriction map  $H_{\mathrm{wR}}^*(X) \rightarrow H_{\mathrm{wR}}^*(X_0)$  is injective, and so is the connecting homomorphism  $H_{\mathrm{wR}}^*(X, X_0) \rightarrow H_{\mathrm{wR}}^*(X_1, X_0)$  of the exact sequence for the triple  $(X, X_1, X_0)$  since  $H_{\mathrm{wR}}^*(X)$  is a free module

over  $H^*(BwR)$  (Proposition 2.13). Combining the exact sequences for the pair  $(X, X_0)$  and the one for triple  $(X, X_1, X_0)$ , we obtain the following exact sequence.

**Proposition 7.4.** *The following sequence is exact:*

$$0 \rightarrow H_{wR}^*(X) \rightarrow H_{wR}^*(X_0) \rightarrow H_{wR}^{*+1}(X_1, X_0)$$

where the middle map is the restriction, and the right map is the connecting homomorphism of the exact sequence for the pair  $(X_1, X_0)$ .

Now, observing that  $\mathbb{P}^1(w_\lambda, w_\mu) \cong \mathbb{CP}^1$  as algebraic varieties, the argument in the proof of Theorem 7.2 in [9] directly applies for the  $wR_{\mathbb{C}}$ -action on  $wGr(d, n)$ , and we obtain Proposition 4.2.

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